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SOLID GEOMETRY

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BY

WEBSTER WELLS, S.B.,

PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS
INSTITUTE OF TECHNOLOGY.

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PREFACE.

IN the revision of the author's work on PLANE AND SOLID GEOMETRY, many important improvements have been effected.

With a class just commencing the study of Geometry, too much emphasis cannot be laid on the *form in which an oral or written demonstration should be presented.*

The beginner requires a certain amount of practice before he can acquire the art of putting a proof in a clear and logical form.

To give this drill, the author has, through the whole of Book I., placed directly after each step in the proof the full statement of the reason, in smaller type, enclosed in brackets.

But too much assistance of this nature is open to serious objections, as it has a tendency to make the pupil a mere automaton, and confirm him in indolent habits of study. It has seemed advisable, therefore, in Books II. to V., inclusive, to give only the number of the section where the required authority is to be found.

The above plan has been submitted to a large number of representative teachers, and in nearly every case has met with the most unqualified approval.

In the Solid Geometry, references are given in full in the first sixteen propositions of Book VI., and by section numbers only through the remainder of the work. On pages ix, x, and xi of the Introduction will be found a few propositions put in a form which is recommended for blackboard work.

Particular attention has been given to the arrangement of the propositions and corollaries in a form for convenient reference. The statement of the corollary has in every case been printed in italics; and in nearly every proposition in which more than one truth is stated, the various parts are distinguished by numerals. Thus, when reference is made to a preceding section, the pupil will readily find the precise statement which is to be quoted.

The exercises are upwards of seven hundred in number, and have been selected with great care. In certain exercises which might otherwise present difficulties to the pupil, reference is made to a previous section or exercise which may be used in the solution. The exercises in each Book are numbered consecutively.

In the Plane Geometry, the new exercises are largely numerical; but in the Solid Geometry, there is a considerable increase in the number of both numerical exercises and original theorems. A number of the exercises are in the nature of alternative methods of proof for preceding propositions.

In the Appendix to the Plane Geometry will be found an additional set of exercises of somewhat greater difficulty than those previously given.

The pages have been arranged in such a way as to avoid the necessity, while reading a proof, of turning the page for reference to the figure.

The attention of teachers is specially invited to the explanations given in the Introduction, commencing on page vii.

The author desires to express his thanks to the many teachers, in all parts of the country, who have furnished him with valuable suggestions and criticisms.

WEBSTER WELLS.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
1894.

CONTENTS.

SOLID GEOMETRY.

BOOK VI.

LINES AND PLANES IN SPACE. — DIEDRALS. — POLYEDRALS.

394. Def. A plane is said to be *determined* by certain lines or points when one plane, and only one, can be drawn through these lines or points.

Proposition I. Theorem.

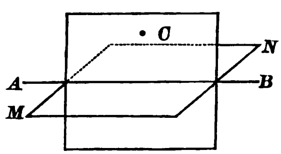
395. *A plane is determined*

I. *By a straight line and a point without the line.*

II. *By three points not in the same straight line.*

III. *By two intersecting straight lines.*

IV. *By two parallel straight lines.*

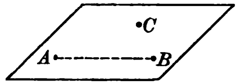


I. Let C be a point without the straight line AB.

To prove that a plane is determined (§ 394) by AB and C.

If any plane, as MN, be drawn through AB, it may be revolved about AB as an axis until it contains the point C.

Hence, one plane, and only one, can be drawn through AB and C.



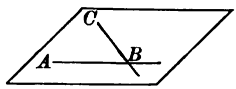
II. Let A, B, and C be three points not in the same straight line.

To prove that a plane is determined by A, B, and C.

Draw AB.

By I., one plane, and only one, can be drawn through the line AB and the point C.

Hence, one plane, and only one, can be drawn through A, B, and C.



III. Let AB and BC be two intersecting straight lines.

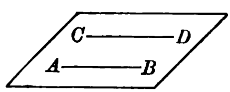
To prove that a plane is determined by AB and BC.

By I., one plane, and only one, can be drawn through AB and any point C of BC.

But since this plane contains the points B and C, it must contain the line BC.

[A plane is a surface such that the straight line joining any two of its points lies entirely in the surface.] (§ 8.)

Hence, one plane, and only one, can be drawn through AB and BC.



IV. Let AB and CD be two parallel lines.

To prove that a plane is determined by AB and CD.

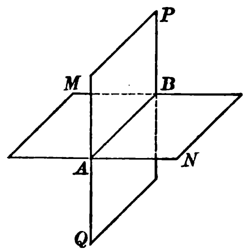
The parallels AB and CD lie in the same plane (§ 52).

And by I., but one plane can be drawn through AB and any point C of CD.

Hence, one plane, and only one, can be drawn through AB and CD.

Proposition II. Theorem.

396. *The intersection of two planes is a straight line.*



Let the line AB be the intersection of the planes MN and PQ.

To prove AB a straight line.

Let a straight line be drawn between the points A and B. This line must lie in MN, and also in PQ.

[A plane is a surface such that the straight line joining any two of its points lies entirely in the surface.] (§ 8.)

Then it must be the intersection of MN and PQ.

Whence, AB is a straight line.

397. Def. If a straight line meets a plane, the point of intersection is called the *foot* of the line.

A straight line is said to be *perpendicular to a plane* when it is perpendicular to every straight line drawn in the plane through its foot.

A straight line is said to be *parallel to a plane* when it cannot meet the plane however far they may be produced.

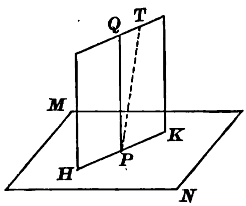
Two planes are said to be *parallel to each other* when they cannot meet however far they may be produced.

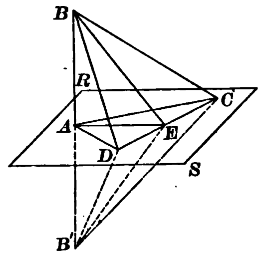
398. Sch. The following form of the second definition of § 397 is given for convenience of reference:

A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.

Proposition III. Theorem.

399. *At a given point in a plane, one perpendicular to the plane can be drawn, and but one.*





Let P be the given point in the plane MN.

To prove that a perpendicular can be drawn to MN at P, and but one.

At any point A of the straight line AB draw the lines AC and AD perpendicular to AB.

Let RS be the plane determined by AC and AD.

Let AE be any other straight line drawn through the point A in the plane RS; and draw the line CED intersecting AC, AE, and AD in C, E, and D.

Produce BA to B', making $AB' = AB$.

Draw BC, BE, BD, $B'C$, $B'E$, and $B'D$.

In the triangles BCD and $B'CD$, the side CD is common.

And since AC and AD are perpendicular to BB' at its middle point,

$$BC = B'C, \text{ and } BD = B'D.$$

[If a perpendicular be erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.] (§ 40, I.)

Whence, $\triangle BCD = \triangle B'CD$.

[Two triangles are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 69.)

Now revolve triangle BCD about CD as an axis until it coincides with triangle $B'CD$.

Then B will fall at B', and the line BE will coincide with $B'E$; that is, $BE = B'E$.

Hence, since the points A and E are each equally distant from B and B', AE is perpendicular to BB'.

[Two points, each equally distant from the extremities of a straight line, determine a perpendicular at its middle point.] (§ 43.)

But AE is *any* straight line drawn through A in RS.

Then, AB is perpendicular to *every* straight line drawn through its foot in the plane RS.

Whence, AB is perpendicular to RS.

[A straight line is said to be perpendicular to a plane when it is perpendicular to every straight line drawn in the plane through its foot.] (§ 397.)

Now apply the plane RS to the plane MN so that the point A shall fall at P; and let AB take the position PQ.

Then, PQ will be perpendicular to MN.

Hence, a perpendicular can be drawn to MN at P.

If possible, let PT be another perpendicular to MN at P; and let the plane determined by PQ and PT intersect MN in the line HK.

Then, both PQ and PT are perpendicular to HK.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

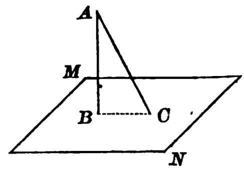
But in the plane HKT, only one perpendicular can be drawn to HK at P.

[At a given point in a straight line, but one perpendicular to the line can be drawn.] (§ 28.)

Hence, but one perpendicular can be drawn to MN at P.

400. Cor. I. *A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.*

401. Cor. II. *From a given point without a plane, one perpendicular to the plane can be drawn, and but one.*



The latter statement is proved as follows:

If possible, let AB and AC be two perpendiculars from A to the plane MN.

Draw BC; then the triangle ABC will have two right angles.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

But this is impossible.

Then but one perpendicular can be drawn from A to MN.

402. Cor. III. *The perpendicular is the shortest line that can be drawn from a point to a plane.*

Let AB be the perpendicular from A to the plane MN, and AC any other straight line from A to MN.

To prove $AB < AC$.

Draw BC; then, since AB is perpendicular to BC,

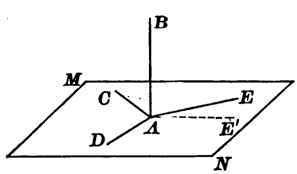
$$AB < AC.$$

[The perpendicular is the shortest line that can be drawn from a point to a straight line.] (§ 45.)

403. Sch. The *distance* of a point from a plane signifies the length of the perpendicular from the point to the plane.

Proposition IV. Theorem.

404. *All the perpendiculars to a straight line at a given point lie in a plane perpendicular to the line.*



Let AC and AD be perpendicular to the line AB at A.

Then the plane MN, determined by AC and AD, is perpendicular to AB.

[A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.] (§ 400.)

Let AE be any other perpendicular to AB at A.

To prove that AE lies in MN.

Let the plane determined by AB and AE intersect MN in AE'; then, AB is perpendicular to AE'.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

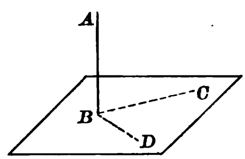
But in the plane ABE, but one perpendicular can be drawn to AB at A.

[At a given point in a straight line, but one perpendicular to the line can be drawn.] (§ 28.)

Then, AE' and AE coincide, and AE lies in the plane MN.

405. Cor. I. *Through a given point in a straight line, a plane can be drawn perpendicular to the line, and but one.*

406. Cor. II. *Through a given point without a straight line, a plane can be drawn perpendicular to the line, and but one.*



Let C be the given point without the straight line AB.

To prove that a plane can be drawn through C perpendicular to AB, and but one.

Draw CB perpendicular to AB, and let BD be any other perpendicular to AB at B.

Then the plane determined by BD and BC will be a plane drawn through C perpendicular to AB.

[A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.] (§ 400.)

But only one perpendicular can be drawn from C to AB.

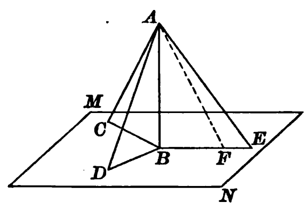
Hence, but one plane can be drawn through C perpendicular to AB.

PROPOSITION V. THEOREM.

407. *If oblique lines be drawn from a point to a plane,*

I. *Two oblique lines cutting off equal distances from the foot of the perpendicular from the point to the plane are equal.*

II. *Of two oblique lines cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.*



I. Let the oblique lines AC and AD meet the plane MN at equal distances from the foot of the perpendicular AB.

To prove $AC = AD$.

Draw BC and BD.

In the triangles ABC and ABD, the side AB is common.

Also, $\angle ABC = \angle ABD$.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

And by hypothesis, $BC = BD$.

Then, $\triangle ABC = \triangle ABD$.

[Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.] (§ 63.)

Whence, $AC = AD$.

[In equal figures, the homologous parts are equal.] (§ 66.)

II. Let the line AE meet MN at a greater distance from B than AC.

To prove $AE > AC$.

Draw BE; on BE take $BF = BC$, and draw AF.

Then, $AF = AC$.

[If oblique lines be drawn from a point to a plane, two oblique lines cutting off equal distances from the foot of the perpendicular from the point to the plane are equal.] (§ 407, I.)

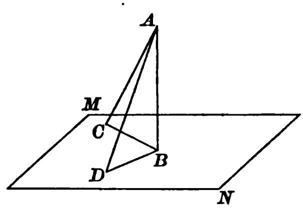
But, $AE > AF$.

[If oblique lines be drawn from a point to a straight line, of two oblique lines cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.] (§ 48, II.)

Whence, $AE > AC$.

PROPOSITION VI. THEOREM.

408. (Converse of Prop. V., I.) *Two equal oblique lines from a point to a plane cut off equal distances from the foot of the perpendicular from the point to the plane.*



Let AC and AD be equal oblique lines, and AB the perpendicular, from A to the plane MN; and draw BC and BD.

To prove $BC = BD$.

In the triangles ABC and ABD, AB is common.

And by hypothesis, $AC = AD$.

Also, ABC and ABD are right angles.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

Whence, $\triangle ABC = \triangle ABD$.

[Two right triangles are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.] (§ 88.)

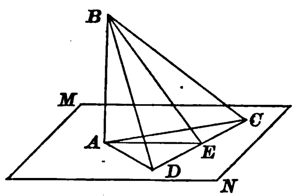
Therefore, $BC = BD$.

409. COR. (Converse of Prop. V., II.) *If two unequal oblique lines be drawn from a point to a plane, the greater cuts off the greater distance from the foot of the perpendicular from the point to the plane.*

(The proof is left to the student.)

PROPOSITION VII. THEOREM.

410. *If through the foot of a perpendicular to a plane a line be drawn at right angles to any line in the plane, the line drawn from its intersection with this line to any point in the perpendicular will be perpendicular to the line in the plane.*



Let AB be perpendicular to the plane MN.

Draw AE perpendicular to any line CD in MN, and join E to any point B in AB.

To prove BE perpendicular to CD.

On CD take $EC = ED$; and draw AC, AD, BC, and BD.

Then, $AC = AD$.

[If a perpendicular be erected at the middle point of a straight line, any point in the perpendicular is equally distant from the extremities of the line.] (§ 40, I.)

Therefore, $BC = BD$.

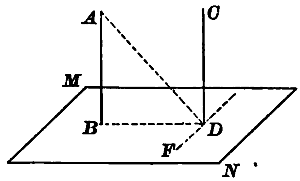
[If oblique lines be drawn from a point to a plane, two oblique lines cutting off equal distances from the foot of the perpendicular from the point to the plane are equal.] (§ 407, I.)

Whence, BE is perpendicular to CD.

[Two points, each equally distant from the extremities of a straight line, determine a perpendicular at its middle point.] (§ 43.)

Proposition VIII. Theorem.

411. *Two perpendiculars to the same plane are parallel.*



Let the lines AB and CD be perpendicular to the plane MN.

To prove AB and CD parallel.

Let A be any point of AB, and draw AD and BD.

Also, draw DF in the plane MN perpendicular to BD.

Then CD is perpendicular to DF.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

Also, AD is perpendicular to DF.

[If through the foot of a perpendicular to a plane a line be drawn at right angles to any line in the plane, the line drawn from its intersection with this line to any point in the perpendicular will be perpendicular to the line in the plane.] (§ 410.)

Therefore, CD, AD, and BD, being perpendicular to DF at D, lie in the same plane.

[All the perpendiculars to a straight line at a given point lie in a plane perpendicular to the line.] (§ 404.)

Hence, AB and CD lie in the same plane.

[A plane is a surface such that the straight line joining any two of its points lies entirely in the surface.] (§ 8.)

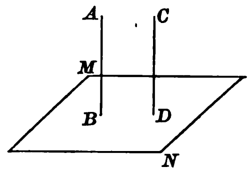
Again, AB and CD are perpendicular to BD.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

Whence, AB and CD are parallel.

[Two perpendiculars to the same straight line are parallel.] (§ 54.)

412. Cor. I. *If one of two parallels is perpendicular to a plane, the other is also perpendicular to the plane.*



Let the lines AB and CD be parallel, and let AB be perpendicular to the plane MN.

To prove CD perpendicular to MN.

A perpendicular from C to MN will be parallel to AB.

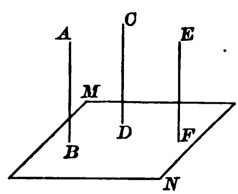
[Two perpendiculars to the same plane are parallel.] (§ 411.)

But through C, only one parallel can be drawn to AB.

[But one straight line can be drawn through a given point parallel to a given straight line.] (§ 53.)

Whence, CD is perpendicular to MN.

413. Cor. II. *If each of two straight lines is parallel to a third, they are parallel to each other.*



Let the lines AB and CD be parallel to EF.

To prove AB and CD parallel.

Draw the plane MN perpendicular to EF.

Then each of the lines AB and CD is perpendicular to MN.

[If one of two parallels is perpendicular to a plane, the other is also perpendicular to the plane.] (§ 412.)

Whence, AB and CD are parallel.

[Two perpendiculars to the same plane are parallel.] (§ 411.)

EXERCISES.

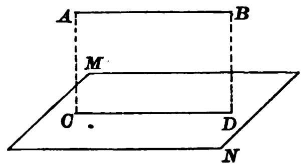
1. What is the locus (§ 141) of the perpendiculars to a given straight line AB at the point A?

2. What is the locus of points equally distant from the circumference of a given circle?

3. If a plane bisects a straight line at right angles, any point in the plane is equally distant from the extremities of the line.

PROPOSITION IX. THEOREM.

414. *A straight line parallel to a line in a plane is parallel to the plane.*



Let AB be parallel to the line CD in the plane MN.

To prove AB parallel to MN.

The parallels AB and CD lie in a plane, which intersects MN in the line CD.

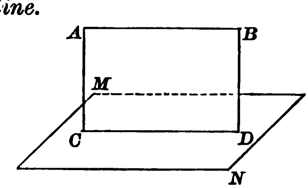
Hence, if AB meets MN, it must be at some point of CD.

But AB, being parallel to CD, cannot meet it.

Then AB and MN cannot meet, and are parallel (§ 397).

PROPOSITION X. THEOREM.

415. *If a straight line is parallel to a plane, the intersection of the plane with any plane drawn through the line is parallel to the line.*



Let the line AB be parallel to the plane MN; and let CD be the intersection of MN with a plane drawn through AB.

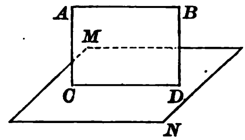
To prove AB and CD parallel.

The lines AB and CD lie in the same plane.

And since AB cannot meet the plane MN however far they may be produced, it cannot meet CD.

Therefore, AB and CD are parallel (§ 52).

416. Cor. *If a line and a plane are parallel, a parallel to the line through any point of the plane lies in the plane.*



Let the line AB be parallel to the plane MN; and through any point C of MN draw CD parallel to AB.

To prove that CD lies in MN.

The plane determined by AB and C intersects MN in a parallel to AB.

[If a straight line is parallel to a plane, the intersection of the plane with any plane drawn through the line is parallel to the line.] (§ 415.)

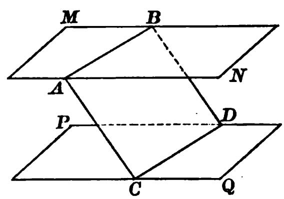
But through C, only one parallel can be drawn to AB.

[But one straight line can be drawn through a given point parallel to a given straight line.] (§ 53.)

Whence, CD lies in MN.

Proposition XI. Theorem.

417. *If two parallel planes are cut by a third plane, the intersections are parallel.*



Let the parallel planes MN and PQ be cut by the plane AD in the lines AB and CD, respectively.

To prove AB and CD parallel.

The lines AB and CD lie in the same plane.

And since the planes MN and PQ cannot meet however far they may be produced, AB and CD cannot meet.

Therefore, AB and CD are parallel (§ 52).

418. Cor. *Parallel lines included between parallel planes are equal.*

Let AC and BD be parallel lines included between the parallel planes MN and PQ.

To prove $AC = BD$.

Let the plane determined by AC and BD intersect MN and PQ in the lines AB and CD.

Then, AB and CD are parallel.

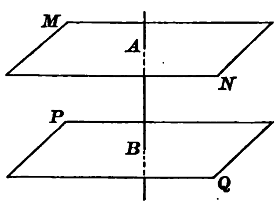
[If two parallel planes are cut by a third plane, the intersections are parallel.] (§ 417.)

Therefore, $AC = BD$.

[Parallel lines included between parallel lines are equal.] (§ 105.)

PROPOSITION XII. THEOREM.

419. *Two planes perpendicular to the same straight line are parallel.*



Let the planes MN and PQ be perpendicular to the line AB.

To prove MN and PQ parallel.

If MN and PQ are not parallel, they will meet if sufficiently produced; let C be a point in their intersection.

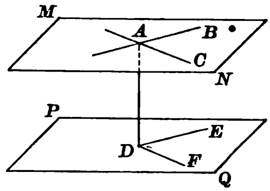
There will then be two planes drawn through C perpendicular to AB, which is impossible.

[Through a given point without a straight line, but one plane can be drawn perpendicular to the line.] (§ 406.)

Whence, MN and PQ cannot meet, and are parallel.

PROPOSITION XIII. THEOREM.

420. *If each of two intersecting lines is parallel to a plane, their plane is parallel to the given plane.*



Let AB and AC be parallel to the plane PQ.
To prove their plane MN parallel to PQ.

Draw AD perpendicular to PQ.
Through D draw DE and DF parallel to AB and AC.
Then, DE and DF lie in the plane PQ.

[If a line and a plane are parallel, a parallel to the line through any point of the plane lies in the plane.] (§ 416.)

Whence, AD is perpendicular to DE and DF.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

Therefore, AD is perpendicular to AB and AC.

[A straight line perpendicular to one of two parallels is perpendicular to the other.] (§ 56.)

Hence, AD is perpendicular to MN.

[A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.] (§ 400.)

Then MN and PQ are parallel.

[Two planes perpendicular to the same straight line are parallel.] (§ 419.)

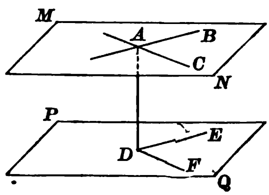
EXERCISES.

4. A line parallel to each of two intersecting planes is parallel to their intersection.

5. A straight line and a plane perpendicular to the same straight line are parallel.

PROPOSITION XIV. THEOREM.

421. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*



Let MN and PQ be parallel planes; and let the line AD be perpendicular to PQ.

To prove AD perpendicular to MN.

Pass any two planes through AD, intersecting MN in AB and AC, and PQ in DE and DF.

Then AB is parallel to DE, and AC to DF.

[If two parallel planes are cut by a third plane, the intersections are parallel.] (§ 417.)

But AD is perpendicular to DE and DF.

[A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.] (§ 398.)

Whence, AD is perpendicular to AB and AC.

[A straight line perpendicular to one of two parallels is perpendicular to the other.] (§ 56.)

Therefore, AD is perpendicular to MN.

[A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.] (§ 400.)

422. COR. I. *Two parallel planes are everywhere equally distant* (§ 403).

For all common perpendiculars to the planes are parallel.

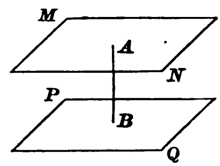
[Two perpendiculars to the same plane are parallel.] (§ 411.)

Therefore they are all equal.

[Parallel lines included between parallel planes are equal.] (§ 418.)

423. COR. II. *Through a given point a plane can be drawn parallel to a given plane, and but one.*

Let A be the given point, and PQ the given plane.



To prove that a plane can be drawn through A parallel to PQ, and but one.

Draw AB perpendicular to PQ.

Through A pass the plane MN perpendicular to AB.

Then MN will be parallel to PQ.

[Two planes perpendicular to the same straight line are parallel.] (§ 419.)

If another plane could be drawn through A parallel to PQ, it would be perpendicular to AB.

[A straight line perpendicular to one of two parallel planes is perpendicular to the other also.] (§ 421.)

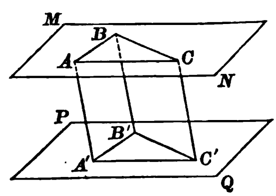
It would then coincide with MN.

[Through a given point in a straight line, but one plane can be drawn perpendicular to the line.] (§ 405.)

Then but one plane can be drawn through A parallel to PQ.

PROPOSITION XV. THEOREM.

424. *If two angles not in the same plane have their sides parallel and extending in the same direction, they are equal, and their planes are parallel.*



Let MN and PQ be the planes of the angles BAC and $B'A'C'$; and let AB and AC be parallel respectively to $A'B'$ and $A'C'$, and extend in the same direction.

I. To prove $\angle BAC = \angle B'A'C'$.

Lay off $AB = A'B'$, and $AC = A'C'$; and draw AA', BB', CC', BC, and $B'C'$.

Then since AB is equal and parallel to $A'B'$, the figure $ABB'A'$ is a parallelogram.

[If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.] (§ 109.)

Whence, AA' is equal and parallel to BB'.

[The opposite sides of a parallelogram are equal.] (§ 104.)

In like manner, AA' is equal and parallel to CC'.

Therefore, BB' is equal and parallel to CC'.

[If each of two straight lines is parallel to a third, they are parallel to each other.] (§ 413.)

Whence, $BB'C'C$ is a parallelogram, and $BC = B'C'$.

Therefore, $\triangle ABC = \triangle A'B'C'$.

[Two triangles are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 69.)

Whence, $\angle BAC = \angle B'A'C'$.

[In equal figures, the homologous parts are equal.] (§ 66.)

II. To prove MN parallel to PQ.

The lines AB and AC are each parallel to the plane PQ.

[A straight line parallel to a line in a plane is parallel to the plane.] (§ 414.)

Therefore, MN is parallel to PQ.

[If each of two intersecting lines is parallel to a plane, their plane is parallel to the given plane.] (§ 420.)

EXERCISES.

6. What is the locus of points equally distant from a given plane?

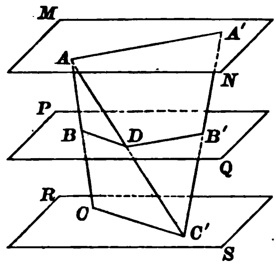
7. If two planes are parallel, a line parallel to one of them through any point of the other lies in the other.

8. If two planes are parallel to a third plane, they are parallel to each other.

9. If a line is parallel to a plane, it is everywhere equally distant from the plane.

PROPOSITION XVI. THEOREM.

425. *If two straight lines are cut by three parallel planes, the corresponding segments are proportional.*



Let the parallel planes MN, PQ, and RS intersect the lines AC and $A'C'$ in the points A, B, C, and A', B', C', respectively.

To prove $$\frac{AB}{BC}=\frac{A'B'}{B'C'}.$$

Draw AC'.

Through AC and AC' pass a plane, intersecting PQ and RS in the lines BD and CC'.

Then BD is parallel to CC'.

[If two parallel planes are cut by a third plane, the intersections are parallel.] (§ 417.)

Therefore, $$\frac{AB}{BC}=\frac{AD}{DC'}. \quad (1)$$

[A parallel to one side of a triangle divides the other two sides proportionally.] (§ 245.)

In like manner, $$\frac{AD}{DC'}=\frac{A'B'}{B'C'}. \quad (2)$$

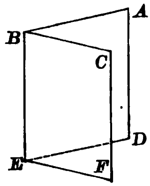
From (1) and (2), $$\frac{AB}{BC}=\frac{A'B'}{B'C'}.$$

Ex. 10. Through a given point a plane can be drawn parallel to any two straight lines in space. (§ 414.)

DIEDRALS.

Definitions.

426. If two planes meet in a straight line, the figure formed is called a *diedral angle,* or simply a *diedral.*



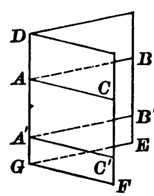
The line of intersection of the planes is called the *edge* of the diedral, and the planes are called its *faces.*

Thus, in the diedral formed by the planes *BD* and *BF*, *BE* is the edge, and *BD* and *BF* are the faces.

427. A diedral may be designated by two letters on its edge; or, if several diedrals have a common edge, by four letters, one in each face and two on the edge, the letters on the edge being named between the other two.

Thus, the above diedral may be designated *BE*, or *ABEC*.

428. The *plane angle* of a diedral is the angle formed by two straight lines drawn one in each face, perpendicular to the edge at the same point.



Thus, if the lines *AB* and *AC* be drawn in the faces *DE* and *DF*, respectively, perpendicular to *DG* at *A*, *BAC* is the plane angle of the diedral *DG*.

429. Let the lines $A'B'$ and $A'C'$ be drawn in the faces *DE* and *DF*, respectively, perpendicular to *DG* at A'.

Then, $A'B'$ is parallel to *AB*, and $A'C'$ to *AC*. (§ 54.)

Whence, $\angle B'A'C' = \angle BAC.$ (§ 424.)

That is, *the plane angle of a diedral is of the same magnitude at whatever point of the edge it may be drawn.*

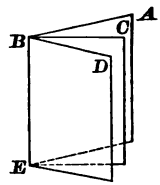
430. Two diedrals are *equal* when their faces may be made to coincide.

431. It is evident that *two diedrals are equal when their plane angles are equal.*

432. Conversely, *the plane angles of equal diedrals are equal.*

433. A plane perpendicular to the edge of a diedral intersects the faces in lines perpendicular to the edge (§ 398).

Hence, *a plane perpendicular to the edge of a diedral intersects the faces in lines which form the plane angle of the diedral.*

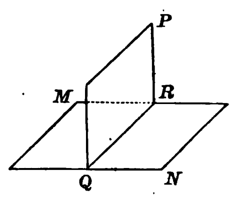


434. Two diedrals are said to be *adjacent* when they have the same edge, and a common face between them; as $ABEC$ and $CBED$.

Two diedrals are said to be *vertical* when the faces of one are the extensions of the faces of the other.

435. Through a given straight line in a plane, a plane may be drawn meeting the given plane in such a way as to make the adjacent diedrals equal. (Compare § 27.)

Each of the equal diedrals is called a *right diedral,* and the planes are said to be *perpendicular* to each other.



Thus, if the plane PQ be drawn meeting the plane MN in such a way as to make the adjacent diedrals $PRQM$ and $PRQN$ equal, each of these is a right diedral, and MN and PQ are perpendicular to each other.

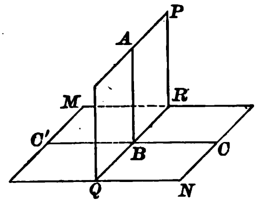
436. Through a given line in a plane but one plane can be drawn perpendicular to the given plane. (Compare § 28.)

437. The *projection of a point on a plane* is the foot of the perpendicular drawn from the point to the plane.

The *projection of a line on a plane* is the locus (§ 141) of the projections of its points.

PROPOSITION XVII. THEOREM.

438. *The plane angle of a right diedral is a right angle.*



Let the planes PQ and MN be perpendicular to each other, and intersect in the line QR.

Let ABC and ABC' be the plane angles of the diedrals $PRQN$ and $PRQM$.

To prove ABC a right angle.

Since PQ is perpendicular to MN, we have

diedral $PRQN$ = diedral $PRQM$. (§ 435.)

Whence, $\angle ABC = \angle ABC'$. (§ 432.)

Therefore, ABC is a right angle. (§ 27.)

439. COR. (Converse of Prop. XVII.) *If the plane angle of a diedral is a right angle, the faces of the diedral are perpendicular to each other.*

Let the planes PQ and MN intersect in the line QR.

Let ABC and ABC' be the plane angles of the diedrals $PRQN$ and $PRQM$, and let ABC be a right angle.

To prove PQ perpendicular to MN.

Since ABC is a right angle, we have

$\angle ABC = \angle ABC'$. (§ 27.)

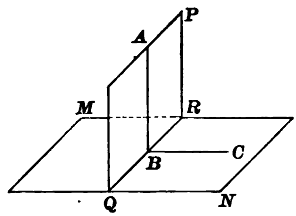
Whence, diedral $PRQN$ = diedral $PRQM$. (§ 431.)

Therefore, PQ is perpendicular to MN. (§ 435.)

Ex. 11. Through any given straight line a plane can be drawn parallel to any other straight line. (§ 414.)

Proposition XVIII. Theorem.

440. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Let the plane PQ be perpendicular to MN.

Let QR be their intersection, and draw AB in the plane PQ perpendicular to QR.

To prove AB perpendicular to MN.

Draw BC in the plane MN perpendicular to QR.

Then ABC is the plane angle of the diedral $PRQN$. (§ 428.)

Whence, ABC is a right angle. (§ 438.)

Therefore AB, being perpendicular to BC and BQ at B, is perpendicular to the plane MN. (§ 400.)

441. Cor. I. *If two planes are perpendicular, a perpendicular to one of them at any point of their intersection lies in the other.*

Let the plane PQ be perpendicular to MN; at any point B in their intersection QR, draw AB perpendicular to MN.

To prove that AB lies in PQ.

A line drawn in PQ perpendicular to QR at B will be perpendicular to MN. (§ 440.)

But at the point B, but one perpendicular can be drawn to MN. (§ 399.)

Hence, AB lies in PQ.

442. Cor. II. *If two planes are perpendicular, a perpendicular to one from any point of the other lies in the other.*

Let the plane PQ be perpendicular to MN; and through any point A of PQ draw AB perpendicular to MN.

To prove that AB lies in PQ.

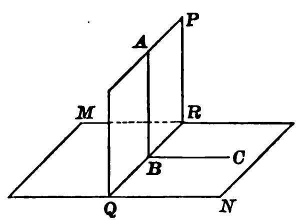
A line drawn in PQ through the point A, perpendicular to the intersection QR, will be perpendicular to MN. (§ 440.)

But from the point A, but one perpendicular can be drawn to MN. (§ 401.)

Hence, AB lies in PQ.

Proposition XIX. Theorem.

443. *If a straight line is perpendicular to a plane, every plane drawn through the line is perpendicular to the plane.*



Let the line AB be perpendicular to the plane MN; and let PQ be any plane drawn through AB.

To prove PQ perpendicular to MN.

Let QR be the intersection of PQ and MN, and draw BC in the plane MN perpendicular to QR.

Now AB is perpendicular to BQ. (§ 398.)

Then ABC is the plane angle of the diedral $PRQN$. (§ 428.)

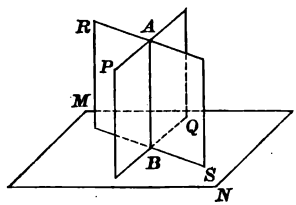
But ABC is a right angle. (§ 398.)

Hence, PQ is perpendicular to MN. (§ 439.)

444. Cor. *A plane perpendicular to the edge of a diedral is perpendicular to its faces.*

Proposition XX. Theorem.

445. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*

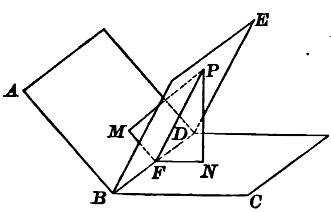


Let the planes PQ and RS be perpendicular to MN.
To prove their intersection AB perpendicular to MN.

Let a perpendicular be drawn to MN at B.
This perpendicular will lie in both PQ and RS. (§ 441.)
It must therefore be their line of intersection.
Hence, AB is perpendicular to MN.

Proposition XXI. Theorem.

446. *Every point in the bisecting plane of a diedral is equally distant from the faces of the diedral.*



From any point P in the bisecting plane BE of the diedral $ABDC$, draw PM and PN perpendicular to AD and CD.

To prove $PM = PN$.

Let the plane determined by PM and PN intersect the planes AD, BE, and CD in FM, FP, and FN.

The plane $PMFN$ is perpendicular to the planes AD and CD. (§ 443.)

Then the plane $PMFN$ is perpendicular to BD. (§ 445.)

Therefore, PFM and PFN are the plane angles of the diedrals $ABDE$ and $CBDE$. (§ 433.)

Whence, $\angle PFM = \angle PFN$. (§ 432.)

Now in the right triangles PFM and PFN, PF is common.

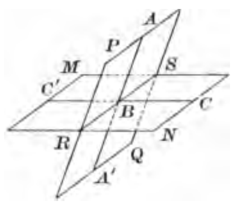
Also, $\angle PFM = \angle PFN$.

Therefore, $\triangle PFM = \triangle PFN$. (§ 70.)

Whence, $PM = PN$. (§ 66.)

PROPOSITION XXII. THEOREM.

447. *If two planes intersect, the vertical diedrals are equal.*



Let the planes MN and PQ intersect in the line RS.

To prove diedral $PRSN$ = diedral $MRSQ$.

Let ABC and $A'BC'$ be the plane angles of the diedrals $PRSN$ and $MRSQ$.

Then, $\angle ABC = \angle A'BC'$. (§ 39.)

Whence, diedral $PRSN$ = diedral $MRSQ$. (§ 431.)

EXERCISES.

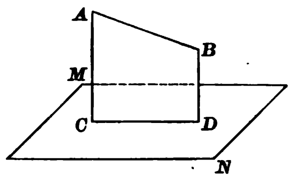
12. If two parallel planes are cut by a third plane, the alternate-interior diedrals are equal.

13. If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

14. If a plane be drawn through a diagonal of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.

Proposition XXIII. Theorem.

448. *Through a given straight line without a plane, a plane can be drawn perpendicular to the given plane, and but one.*



Let AB be the given line without the plane MN.

To prove that a plane can be drawn through AB perpendicular to MN, and but one.

Draw AC perpendicular to MN, and let AD be the plane determined by AB and AC.

Then, AD is perpendicular to MN. (§ 443.)

If more than one plane could be drawn through AB perpendicular to MN, their common intersection, AB, would be perpendicular to MN. (§ 445.)

Hence, but one plane can be drawn through AB perpendicular to MN, unless AB is perpendicular to MN.

Note. If the line AB is perpendicular to MN, an indefinitely great number of planes can be drawn through AB perpendicular to MN (§ 443).

449. Cor. *The projection of a straight line on a plane is a straight line.*

Let CD be the projection of the straight line AB on the plane MN.

To prove CD a straight line.

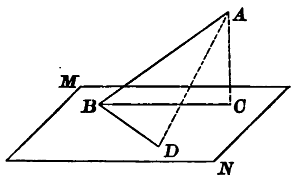
Let a plane be drawn through AB perpendicular to MN.

The perpendiculars to MN from all points of AB will lie in this plane. (§ 442.)

Therefore, CD is a straight line. (§ 396.)

Proposition XXIV. Theorem

450. *The angle between a straight line and its projection on a plane is the least angle which it makes with any line drawn in the plane through its foot.*



Let BC be the projection of the line AB on the plane MN.
Let BD be any other line drawn through B in MN.
To prove $\angle ABC < \angle ABD$.

Lay off $BD = BC$, and draw AC and AD.
Then in the triangles ABC and ABD, AB is common.
Also, $AC < AD$. (§ 402.)
Whence, $\angle ABC < \angle ABD$. (§ 90.)

451. Sch. ABC is called the *angle* between AB and MN.

EXERCISES.

15. If two parallels meet a plane, they make equal angles with it.

16. If a straight line intersects two parallel planes, it makes equal angles with them.

17. The angle between perpendiculars to the faces of a diedral from any point within the angle is the supplement of its plane angle.

18. If BC is the projection of the line AB upon the plane MN, and BD and BE be drawn in the plane making $\angle CBD = \angle CBE$, prove that $\angle ABD = \angle ABE$.

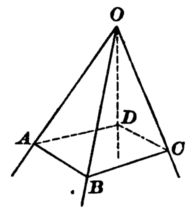
19. If each of two intersecting planes be cut by two parallel planes, not parallel to their intersection, their intersections with the parallel planes include equal angles.

20. The line AB is perpendicular to the plane MN at B. A line is drawn from B meeting the line CD of the plane MN at E. If AE is perpendicular to CD, prove that BE is perpendicular to CD.

POLYEDRALS.

Definitions.

452. If three or more planes meet in a common point, the figure formed is called a *polyedral angle*, or simply a *polyedral.*



The common point is called the *vertex* of the polyedral, and the intersections of the planes the *edges.*

The portions of the planes included between the edges are called the *faces* of the polyedral, and the angles formed by the edges are called the *face angles.*

Thus, in the polyedral O–$ABCD$, O is the vertex; OA, OB, etc., are the edges; the planes AOB, BOC, etc., are the faces; and the angles AOB, BOC, etc., are the face angles.

453. A polyedral must have at least three faces.

A polyedral of three faces is called a *triedral.*

454. To show more distinctly the relative positions of the edges of a polyedral, it is customary to represent them as intersected by a plane, as shown in the figure of § 452.

The plane $ABCD$ is called the *base* of the polyedral.

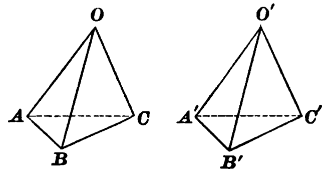
455. The polyedral is not regarded as limited by the base; thus, the face AOB is understood to mean, not the triangle AOB, but the indefinite plane included between the edges OA and OB produced indefinitely.

456. A polyedral is called *convex* when its base is a convex polygon (§ 120).

457. Two polyedrals are called *vertical* when the edges of one are the prolongations of the edges of the other.

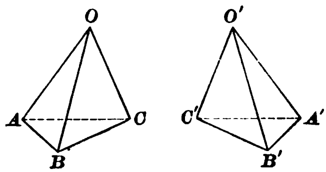
458. Two polyedrals are *equal* when they can be applied to each other so that their faces shall coincide.

459. Two polyedrals are equal when the face angles and diedrals of one are equal respectively to the homologous face angles and diedrals of the other, if the equal parts are arranged *in the same order*.



Thus, if the face angles AOB, BOC, and COA are equal respectively to the face angles $A'O'B'$, $B'O'C'$, and $C'O'A'$, and the diedrals OA, OB, and OC to the diedrals $O'A'$, $O'B'$, and $O'C'$, the triedrals $O\text{–}ABC$ and $O'\text{–}A'B'C'$ are equal; for they can evidently be applied to each other so that their faces shall coincide.

460. Two polyedrals are said to be *symmetrical* when the face angles and diedrals of one are equal respectively to the homologous face angles and diedrals of the other, if the equal parts are arranged *in the reverse order*.



Thus, if the face angles AOB, BOC, and COA are equal respectively to the face angles $A'O'B'$, $B'O'C'$, and $C'O'A'$, and the diedrals OA, OB, and OC to the diedrals $O'A'$, $O'B'$, and $O'C'$, the triedrals $O\text{–}ABC$ and $O'\text{–}A'B'C'$ are symmetrical.

461. It is evident that, in general, two symmetrical polyedrals cannot be placed so that their faces shall coincide.

EXERCISES.

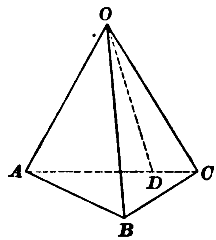
21. The three planes bisecting the diedrals of a triedral meet in a common straight line.

22. D is any point in the perpendicular AF from A to the side BC of the triangle ABC. If DE be drawn perpendicular to the plane of ABC, and GH be drawn through E parallel to BC, prove that AE is perpendicular to GH. (§ 398.)

Proposition XXV. Theorem.

462. *The sum of any two face angles of a triedral is greater than the third.*

Note. The theorem requires proof only in the case where the third angle is greater than either of the others.



In the triedral $O\text{–}ABC$, let the face angle AOC be greater than either AOB or BOC.

To prove $\angle AOB + \angle BOC > \angle AOC.$

In the face AOC, draw the line OD equal to OB, making $\angle AOD = \angle AOB$; and through B and D pass a plane cutting the faces of the triedral in AB, BC, and CA.

Then in the triangles AOB and AOD, OA is common.

And by construction, $OB = OD$,

and $\angle AOB = \angle AOD$.

Therefore, $\triangle AOB = \triangle AOD$. (§ 63.)

Whence, $AB = AD$. (§ 66.)

Now, $AB + BC > AD + DC$. (Ax. 6.)

Or, since $AB = AD$, $BC > DC$.

Then in the triangles BOC and COD, OC is common.

Also, $OB = OD$, and $BC > CD$.

Whence, $\angle BOC > \angle COD$. (§ 90.)

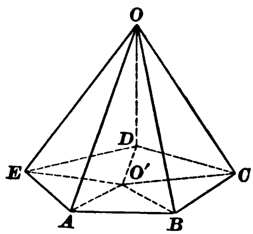
Adding $\angle AOB$ to the first member of this inequality, and its equal $\angle AOD$ to the second member, we have

$$\angle AOB + \angle BOC > \angle AOD + \angle COD.$$

Whence, $\angle AOB + \angle BOC > \angle AOC$.

Proposition XXVI. Theorem.

463. *The sum of the face angles of any convex polyedral is less than four right angles.*



Let $O\text{–}ABCDE$ be a convex polyedral.

To prove $\angle AOB + \angle BOC +$ etc. $<$ four right angles.

Let $ABCDE$ be the base of the polyedral.

Let O' be any point within the polygon $ABCDE$, and draw $O'A$, $O'B$, $O'C$, $O'D$, and $O'E$.

Then, $\angle OAE + \angle OAB > \angle O'AE + \angle O'AB$. (§ 462.)

In like manner,

$$\angle OBA + \angle OBC > \angle O'BA + \angle O'BC; \text{ etc.}$$

Adding these inequalities, we have the sum of the angles at the bases of the triangles whose common vertex is O greater than the sum of the angles at the bases of the triangles whose common vertex is O'.

But the sum of *all* the angles of the triangles whose common vertex is O is equal to the sum of *all* the angles of the triangles whose common vertex is O'. (§ 82.)

Hence, the sum of the angles at O is less than the sum of the angles at O'.

Therefore, the sum of the angles at O is less than four right angles. (§ 37.)

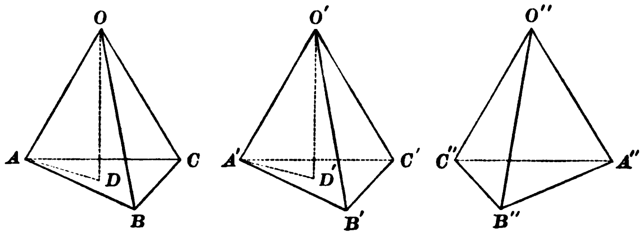
Ex. 23. Between two straight lines not in the same plane a common perpendicular can be drawn. (Ex. 11.)

Proposition XXVII. Theorem.

464. *If two triedrals have the face angles of one equal respectively to the face angles of the other,*

I. *They are equal if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in the reverse order.*



I. In the triedrals $O\text{–}ABC$ and $O'\text{–}A'B'C'$, let

$$\angle AOB = \angle A'O'B',\ \angle BOC = \angle B'O'C',$$
$$\text{and } \angle COA = \angle C'O'A'.$$

To prove triedral $O\text{–}ABC$ = triedral $O'\text{–}A'B'C'$.

Lay off the six equal distances OA, OB, OC, $O'A'$, $O'B'$, and $O'C'$; and draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

Then, $\triangle OAB = \triangle O'A'B'$. (§ 63.)

Whence, $AB = A'B'$. (§ 66.)

Similarly, $BC = B'C'$, and $CA = C'A'$.

Therefore, $\triangle ABC = \triangle A'B'C'$. (§ 69.)

Draw OD and $O'D'$ perpendicular to ABC and $A'B'C'$, respectively; also, draw AD and $A'D'$.

The equal oblique lines OA, OB, and OC meet the plane ABC at equal distances from D. (§ 408.)

Hence, D is the centre of the circumscribed circle of the triangle ABC; and similarly, D' is the centre of the circumscribed circle of $A'B'C'$.

Now apply $O'\text{–}A'B'C'$ to $O\text{–}ABC$, so that the points A', B', and C' shall fall at A, B, and C, and the point D' at D.

Then the perpendicular $O'D'$ will fall upon OD. (§ 399.)

But the right triangles OAD and $O'A'D'$ are equal. (§ 88.)

Whence, $O'D' = OD$, and the point O' will fall at O.

Therefore, the triedrals $O\text{–}ABC$ and $O'\text{–}A'B'C'$ coincide throughout, and are equal.

II. In the triedrals $O\text{–}ABC$ and $O''\text{–}A''B''C''$, let the angles AOB, BOC, and COA be equal respectively to $A''O''B''$, $B''O''C''$, and $C''O''A''$.

To prove $O\text{–}ABC$ symmetrical to $O''\text{–}A''B''C''$.

Construct $O'\text{–}A'B'C'$ symmetrical to $O''\text{–}A''B''C''$, having the angles $A'O'B'$, $B'O'C'$, and $C'O'A'$ equal respectively to $A''O''B''$, $B''O''C''$, and $C''O''A''$.

Then the triedrals $O\text{–}ABC$ and $O'\text{–}A'B'C'$ have the angles AOB, BOC, and COA equal respectively to $A'O'B'$, $B'O'C'$, and $C'O'A'$.

Hence, triedral $O\text{–}ABC$ = triedral $O'\text{–}A'B'C'$. (§ 464, I.)

Therefore, $O\text{–}ABC$ is symmetrical to $O''\text{–}A''B''C''$.

465. COR. *If two triedrals have the face angles of one equal respectively to the face angles of the other, their homologous diedrals are equal.*

EXERCISES.

24. **Two triedrals are equal when two face angles and the included diedral of one are equal respectively to two face angles and the included diedral of the other, and similarly arranged.**

25. **Two triedrals are equal when a face angle and the adjacent diedrals of one are equal respectively to a face angle and the adjacent diedrals of the other, and similarly arranged.**

26. **A is any point in the face EG of the diedral $DEFG$. If AC be drawn perpendicular to the edge EF, and AB perpendicular to the face DF, prove that the plane determined by AC and BC is perpendicular to EF.**

27. **From any point E within the diedral $CABD$, EF and EG are drawn perpendicular to the faces ABC and ABD, and GH perpendicular to the face ABC at H. Prove FH perpendicular to AB.**

BOOK VII.

POLYEDRONS.

DEFINITIONS.

466. A *polyedron* is a solid bounded by planes.

The bounding planes are called the *faces* of the polyedron; their intersections are called the *edges*, and the intersections of the edges the *vertices*.

A *diagonal* is a straight line joining any two vertices not in the same face.

467. The least number of planes which can form a polyedral is three (§ 453); hence, the least number of planes which can bound a polyedron is four.

A polyedron of four faces is called a *tetraedron*; of six faces, a *hexaedron*; of eight faces, an *octaedron*; of twelve faces, a *dodecaedron*; of twenty faces, an *icosaedron*.

468. A polyedron is called *convex* when the section made by any plane is a convex polygon (§ 120).

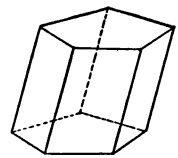
All polyedrons considered hereafter will be understood to be convex.

469. The *volume* of a solid is its ratio to another solid, called the *unit of volume*, adopted arbitrarily as the unit of measure (§ 179).

470. Two solids are said to be *equivalent* when their volumes are equal.

PRISMS AND PARALLELOPIPEDS.

471. A *prism* is a polyedron, two of whose faces are equal polygons lying in parallel planes, having their homologous sides parallel, the other faces being parallelograms.



The equal and parallel faces are called the *bases* of the prism, and the remaining faces the *lateral faces*; the intersections of the lateral faces are called the *lateral edges*, and the sum of the areas of the lateral faces the *lateral area.*

The *altitude* is the perpendicular distance between the planes of the bases.

472. The following is given for convenience of reference:

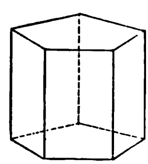
The bases of a prism are equal.

473. It follows from the definition of § 471 that

The lateral edges of a prism are equal and parallel.

474. A prism is called *triangular, quadrangular,* etc., according as its base is a triangle, quadrilateral, etc.

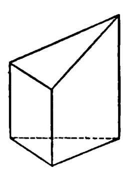
475. A *right prism* is a prism whose lateral edges are perpendicular to its bases.



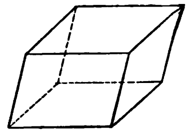
An *oblique prism* is a prism whose lateral edges are not perpendicular to its bases.

476. A *regular prism* is a right prism whose base is a regular polygon.

477. A *truncated prism* is that portion of a prism included between the base, and a plane, not parallel to the base, cutting all the lateral edges.

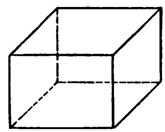


478. A *right section* of a prism is the section made by a plane perpendicular to the lateral edges.



479. A *parallelopiped* is a prism whose bases are parallelograms; that is, all the faces are parallelograms.

480. A *right parallelopiped* is a parallelopiped whose lateral edges are perpendicular to its bases.



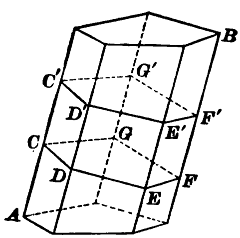
481. A *rectangular parallelopiped* is a right parallelopiped whose bases are rectangles; that is, all the faces are rectangles.

The *dimensions* are the three edges which meet at any vertex.

482. A *cube* is a rectangular parallelopiped whose six faces are all squares.

PROPOSITION I. THEOREM.

483. *The sections of a prism made by two parallel planes which cut all the lateral edges, are equal polygons.*



Let the parallel planes CF and $C'F'$ cut all the lateral edges of the prism AB.

To prove that the sections $CDEFG$ and $C'D'E'F'G'$ are equal.

We have CD parallel to $C'D'$, DE to $D'E'$, etc. (§ 417.)

Whence, $CD = C'D'$, $DE = D'E'$, etc. (§ 105.)

Then the polygons $CDEFG$ and $C'D'E'F'G'$ are mutually equilateral.

Again, $\angle CDE = \angle C'D'E'$,

$\angle DEF = \angle D'E'F'$, etc. (§ 424.)

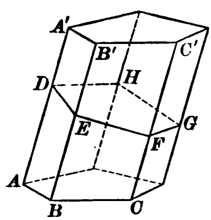
Then the polygons $CDEFG$ and $C'D'E'F'G'$ are mutually equiangular.

Therefore, $CDEFG$ and $C'D'E'F'G'$ are equal. (§ 124.)

484. Cor. *The section of a prism made by a plane parallel to the base is equal to the base.*

Proposition II. Theorem.

485. *The lateral area of a prism is equal to the perimeter of a right section multiplied by a lateral edge.*



Let $DEFGH$ be a right section of the prism AC'.

To prove

lat. area $AC' = (DE + EF + \text{etc.}) \times AA'$.

We have DE perpendicular to AA'. (§ 398.)

Whence, area $AA'B'B = DE \times AA'$. (§ 310.)

Similarly, area $BB'C'C = EF \times BB'$

$= EF \times AA'$; etc. (§ 473.)

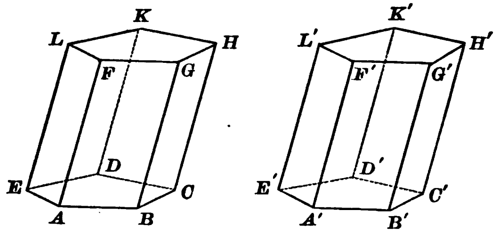
Adding these equations, we have

$$\text{lat. area } AC' = DE \times AA' + EF \times AA' + \text{etc.}$$
$$= (DE + EF + \text{etc.}) \times AA'.$$

486. Cor. *The lateral area of a right prism is equal to the perimeter of the base multiplied by the altitude.*

PROPOSITION III. THEOREM.

487. *Two prisms are equal when the faces including a triedral of one are equal respectively to the faces including a triedral of the other, and similarly placed.*



In the prisms AH and $A'H'$, let the faces $ABCDE$, AG, and AL be equal respectively to the faces $A'B'C'D'E'$, $A'G'$, and $A'L'$; the equal parts being similarly placed.

To prove the prisms equal.

The angles EAB, EAF, and FAB are equal respectively to the angles $E'A'B'$, $E'A'F'$, and $F'A'B'$.

Then, triedral $A\text{-}BEF =$ triedral $A'\text{-}B'E'F'$. (§ 464, I.)

Then the prism $A'H'$ may be applied to AH so that the vertices A', B', C', D', E', G', F', and L' shall fall at A, B, C, D, E, G, F, and L, respectively.

Now since the lateral edges of the prisms are parallel, the edge $C'H'$ will fall upon CH, and $D'K'$ upon DK.

And since the points G', F', and L' fall at G, F, and L, the planes of the upper bases will coincide. (§ 395, II.)

Therefore, the points H' and K' fall at H and K.

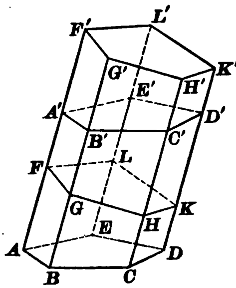
Hence, the prisms coincide throughout, and are equal.

488. SCH. The above demonstration applies without change to the case of two *truncated prisms*.

489. COR. *Two right prisms are equal when they have equal bases and equal altitudes;* for by inverting one of the prisms if necessary, the equal faces will be similarly placed.

PROPOSITION IV. THEOREM.

490. *An oblique prism is equivalent to a right prism, having for its base a right section of the oblique prism, and for its altitude a lateral edge of the oblique prism.*



Let $FGHKL$ be a right section of the oblique prism AD'.

Produce AA' to F', making $FF' = AA'$.

At F' pass the plane $F'K'$ parallel to $FGHKL$, meeting the edges BB', CC', etc., produced at G', H', etc.

To prove AD' equivalent to the right prism FK'.

In the truncated prisms AK and $A'K'$, the faces $FGHKL$ and $F'G'H'K'L'$ are equal. (§ 472.)

Therefore, $A'K'$ may be applied to AK so that the vertices F', G', etc., shall fall at F, G, etc., respectively.

Then the edges $A'F'$, $B'G'$, etc., will coincide in direction with AF, BG, etc. (§ 399.)

But, since $FF' = AA'$, we have $AF = A'F'$.

In like manner, $BG = B'G'$, $CH = C'H'$, etc.

Hence, the vertices A', B', etc., will fall at A, B, etc.

Then, $A'K'$ and AK coincide throughout, and are equal.

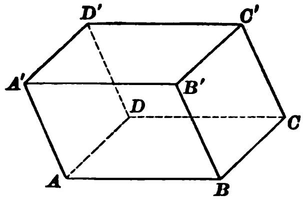
Now taking from the entire solid AK' the truncated prism $A'K'$, there remains the prism AD'.

And taking its equal AK, there remains the prism FK'.

Hence, AD' and FK' are equivalent.

Proposition V. Theorem.

491. *The opposite faces of a parallelopiped are equal and parallel.*



Let AC and $A'C'$ be the bases of the parallelopiped AC'.

To prove the faces AB' and DC' equal and parallel.

AB is equal and parallel to DC, and AA' to DD'. (§ 104.)

Hence, $\angle A'AB = \angle D'DC$,

and the faces AB' and DC' are parallel. (§ 424.)

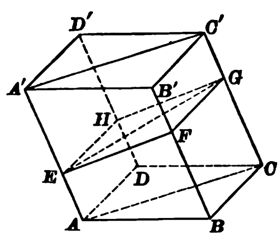
Therefore, the faces AB' and DC' are equal. (§ 112.)

In like manner, we may prove AD' and BC' equal and parallel.

492. Cor. *Either face of a parallelopiped may be taken as the base.*

Proposition VI. Theorem.

493. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*



Let AC' be a parallelopiped.

Through the edges AA' and CC' pass a plane dividing AC' into two triangular prisms, $ABC\text{–}A'$ and $ACD\text{–}A'$.

To prove $ABC\text{–}A' \Leftrightarrow ACD\text{–}A'$.

Let $EFGH$ be a right section of the parallelopiped, cutting the plane $AA'C'C$ in EG.

Now the planes AB' and DC' are parallel. (§ 491.)

Whence, EF is parallel to GH. (§ 417.)

In like manner, EH is parallel to FG.

Therefore, $EFGH$ is a parallelogram.

Whence, $\triangle EFG = \triangle EGH$. (§ 106.)

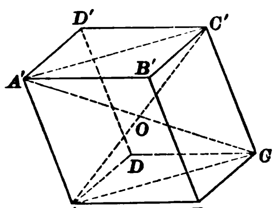
Now, $ABC\text{–}A'$ is equivalent to a right prism whose base is EFG, and altitude AA'; and $ACD\text{–}A'$ is equivalent to a right prism whose base is EGH, and altitude AA'. (§ 490.)

But these two right prisms are equal. (§ 489.)

Therefore, $ABC\text{–}A' \Leftrightarrow ACD\text{–}A'$.

Proposition VII. Theorem.

494. *The diagonals of a parallelopiped bisect each other.*



Let AC' and $A'C$ be diagonals of the parallelopiped AC'.

To prove that AC' and $A'C$ bisect each other.

Draw AC and $A'C'$.

Then AA' is equal and parallel to CC'. (§ 473.)

Whence, the figure $AA'C'C$ is a parallelogram. (§ 109.)

Therefore, AC' and $A'C$ bisect each other at O. (§ 110.)

In like manner, we may prove that any two of the four diagonals AC', $A'C$, BD', and $B'D$ bisect each other at O.

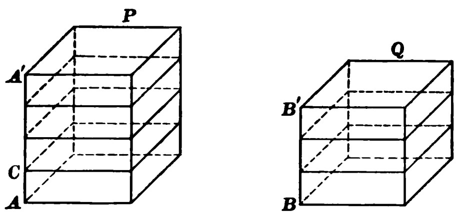
Note. The point O is called the *centre* of the parallelopiped.

Proposition VIII. Theorem.

495. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*

Note. The phrase "rectangular parallelopiped" in the above statement signifies the *volume* of the rectangular parallelopiped.

Case I. *When the altitudes are commensurable.*



Let P and Q be two rectangular parallelopipeds, having equal bases, and commensurable altitudes AA' and BB'.

To prove $$\frac{P}{Q} = \frac{AA'}{BB'}.$$

Let AC be a common measure of AA' and BB', and let it be contained 4 times in AA', and 3 times in BB'.

Then, $$\frac{AA'}{BB'} = \frac{4}{3}. \qquad (1)$$

At the several points of division of AA' and BB' pass planes perpendicular to these lines.

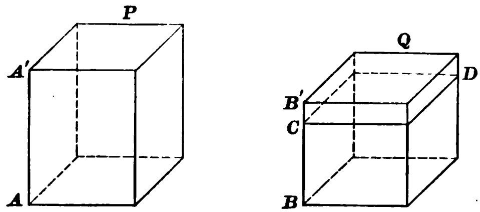
Then the parallelopiped P will be divided into 4 parts, and the parallelopiped Q into 3 parts, all of which parts will be equal. (§ 489.)

Therefore, $$\frac{P}{Q} = \frac{4}{3}. \qquad (2)$$

From (1) and (2), we have

$$\frac{P}{Q} = \frac{AA'}{BB'}.$$

CASE II. *When the altitudes are incommensurable.*



Let P and Q be two rectangular parallelopipeds, having equal bases, and incommensurable altitudes AA' and BB'.

To prove $$\frac{P}{Q} = \frac{AA'}{BB'}.$$

Let AA' be divided into any number of equal parts, and let one of these parts be applied to BB' as a measure.

Since AA' and BB' are incommensurable, a certain number of the parts will extend from B to C, leaving a remainder CB' less than one of the parts.

Pass the plane CD perpendicular to BB', and let Q' denote the rectangular parallelopiped BD.

Then since AA' and BC are commensurable,

$$\frac{P}{Q'} = \frac{AA'}{BC}. \qquad \text{(§ 495, Case I.)}$$

Now let the number of subdivisions of AA' be indefinitely increased.

Then the length of each part will be indefinitely diminished, and the remainder CB' will approach the limit 0.

Then, $\frac{P}{Q'}$ will approach the limit $\frac{P}{Q}$,

and $\frac{AA'}{BC}$ will approach the limit $\frac{AA'}{BB'}$.

By the Theorem of Limits, these limits are equal. (§ 188.)

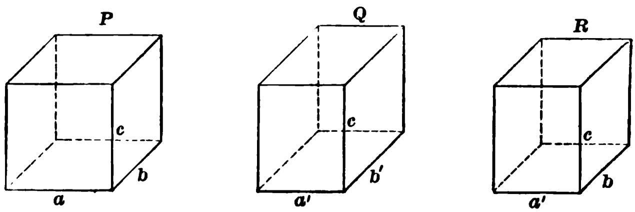
Whence, $$\frac{P}{Q} = \frac{AA'}{BB'}.$$

496. Sch. The theorem may also be expressed:

Two rectangular parallelopipeds having two dimensions in common, are to each other as their third dimensions.

Proposition IX. Theorem.

497. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Let P and Q be two rectangular parallelopipeds, having the dimensions a, b, c, and a', b', c, respectively.

To prove $$\frac{P}{Q} = \frac{a \times b}{a' \times b'}.$$

Let R be a rectangular parallelopiped having the dimensions a', b, and c.

Then P and R have the dimensions b and c in common.

Whence, $$\frac{P}{R} = \frac{a}{a'}. \qquad (\S\ 496.)$$

And R and Q have the dimensions a' and c in common.

Whence, $$\frac{R}{Q} = \frac{b}{b'}.$$

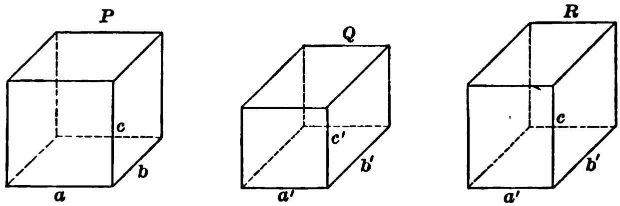
Multiplying these equations, we have
$$\frac{P}{Q} = \frac{a \times b}{a' \times b'}.$$

498. Sch. The theorem may also be expressed:

Two rectangular parallelopipeds having one dimension in common, are to each other as the products of their other two dimensions.

PROPOSITION X. THEOREM.

499. *Any two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Let P and Q be two rectangular parallelopipeds, having the dimensions a, b, c, and a', b', c', respectively.

To prove $$\frac{P}{Q} = \frac{a \times b \times c}{a' \times b' \times c'}.$$

Let R be a rectangular parallelopiped having the dimensions a', b', and c.

Then P and R have the dimension c in common.

Whence, $$\frac{P}{R} = \frac{a \times b}{a' \times b'}. \qquad (\S\ 498.)$$

And R and Q have the dimensions a' and b' in common.

Whence, $$\frac{R}{Q} = \frac{c}{c'}. \qquad (\S\ 496.)$$

Multiplying these equations, we have
$$\frac{P}{Q} = \frac{a \times b \times c}{a' \times b' \times c'}.$$

EXERCISES.

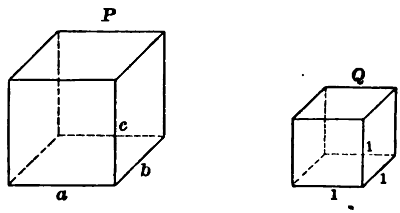
1. Two rectangular parallelopipeds have the dimensions 6, 8, and 14, and 7, 8, and 9, respectively. What is the ratio of their volumes?

2. Find the ratio of the volumes of two rectangular parallelopipeds, whose dimensions are 8, 12, and 21, and 14, 15, and 24, respectively.

3. The diagonals of a rectangular parallelopiped are equal,

PROPOSITION XI. THEOREM.

500. *If the unit of volume is the cube whose edge is the linear unit, the volume of a rectangular parallelopiped is equal to the product of its three dimensions.*



Let a, b, and c be the dimensions of the rectangular parallelopiped P; and let Q be the unit of volume, i.e., a cube whose edge is the linear unit.

To prove vol. $P = a \times b \times c$.

We have $\dfrac{P}{Q} = \dfrac{a \times b \times c}{1 \times 1 \times 1} = a \times b \times c.$ (§ 499.)

But since Q is the unit of volume,

$$\frac{P}{Q} = \text{vol. } P. \qquad \text{(§ 469.)}$$

Whence, vol. $P = a \times b \times c$.

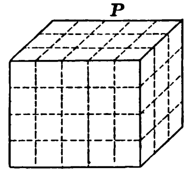
501. COR. I. *The volume of a cube is equal to the cube of its edge.*

502. COR. II. If c be taken as the altitude of the parallelopiped P, $a \times b$ is the area of its base. (§ 305.)

Hence, *the volume of a rectangular parallelopiped is equal to the product of its base and altitude.*

503. SCH. I. In all succeeding theorems relating to volumes, it is understood that the *unit of volume* is the cube whose edge is the linear unit, and the *unit of surface* the square whose side is the linear unit. (Compare § 307.)

504. Sch. II. If the dimensions of the rectangular parallelopiped are *multiples* of the linear unit, the truth of Prop. XI. may be seen by dividing the solid into cubes, each equal to the unit of volume.



Thus, if the dimensions of the rectangular parallelopiped P are 5 units, 4 units, and 3 units, respectively, the solid can evidently be divided into 60 cubes.

In this case, 60, the number which expresses the volume of the rectangular parallelopiped, is the product of 5, 4, and 3, the numbers which express the lengths of its edges.

EXERCISES.

4. Find the altitude of a rectangular parallelopiped, the dimensions of whose base are 21 and 30, equivalent to a rectangular parallelopiped whose dimensions are 27, 28, and 35.

5. Find the edge of a cube equivalent to a rectangular parallelopiped whose dimensions are 9 in., 1 ft. 9 in., and 4 ft. 1 in.

6. Find the volume, and the area of the entire surface, of a cube whose edge is $3\frac{1}{4}$ in.

7. Find the area of the entire surface of a rectangular parallelopiped, the dimensions of whose base are 11 and 13, and volume 858.

8. Find the volume of a rectangular parallelopiped, the dimensions of whose base are 14 and 9, and the area of whose entire surface is 620.

9. Find the dimensions of the base of a rectangular parallelopiped, the area of whose entire surface is 320, volume 336, and altitude 4.

10. How many bricks, each 8 in. long, $2\frac{3}{4}$ in. wide, and 2 in. thick, will be required to build a wall 18 ft. long, 3 ft. high, and 11 in. thick?

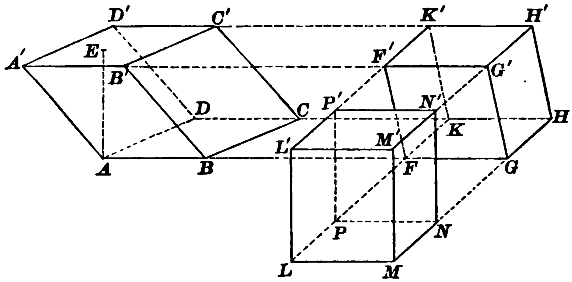
11. The section of a prism made by a plane parallel to a lateral edge is a parallelogram.

12. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of its dimensions.

13. Find the length of the diagonal of a rectangular parallelopiped whose dimensions are 8, 9, and 12.

PROPOSITION XII. THEOREM.

505. *The volume of any parallelopiped is equal to the product of its base and altitude.*



Let AE be the altitude of the parallelopiped AC'.

To prove vol. $AC' = ABCD \times AE$.

Produce the edges AB, $A'B'$, $D'C'$, and DC.

On AB produced, take $FG = AB$; and pass the planes FK' and GH' perpendicular to FG, forming the right parallelopiped FH'.

Then, FH' is equivalent to AC'. (§ 490.)

Produce the edges HG, $H'G'$, $K'F'$, and KF.

On HG produced, take $NM = HG$; and pass the planes NP' and ML' perpendicular to NM, forming the right parallelopiped LN'.

Then, LN' is equivalent to FH'. (§ 490.)

Whence, LN' is equivalent to AC'.

Now since FG is perpendicular to the plane GH', the planes LH and MH' are perpendicular. (§ 443.)

But LMM' is the plane angle of the diedral $LMHH'$. (§ 433.)

Whence, LMM' is a right angle. (§ 438.)

Therefore, LM' is a rectangle, and LN' is a rectangular parallelopiped.

Whence, vol. $LN' = LMNP \times MM'$. (§ 502.)

That is, vol. $AC' = LMNP \times MM'$. (1)

But the rectangle $LMNP$ is equal to the rectangle $FGHK$; for they have equal bases MN and GH, and the same altitude. (§ 113.)

And the rectangle $FGHK$ is equivalent to the parallelogram $ABCD$; for they have equal bases FG and AB, and the same altitude. (§ 311.)

Therefore, $LMNP$ is equivalent to $ABCD$.

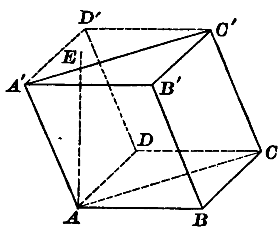
Again, $MM' = AE$. (§ 422.)

Substituting these values in (1), we have

$$\text{vol. } AC' = ABCD \times AE.$$

Proposition XIII. Theorem.

506. *The volume of a triangular prism is equal to the product of its base and altitude.*



Let AE be the altitude of the triangular prism $ABC\text{–}C'$.

To prove vol. $ABC\text{–}C' = ABC \times AE$.

Construct the parallelopiped $ABCD\text{–}D'$, having its edges parallel to AB, BC, and BB', respectively.

Then, vol. $ABC\text{–}C' = \frac{1}{2}$ vol. $ABCD\text{–}D'$ (§ 493.)

$= \frac{1}{2} ABCD \times AE$ (§ 505.)

$= ABC \times AE$. (§ 106.)

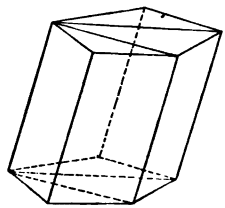
EXERCISES.

14. Find the lateral area and volume of a regular triangular prism, each side of whose base is 5, and whose altitude is 8.

15. The diagonal of a cube is equal to its edge multiplied by $\sqrt{3}$.

PROPOSITION XIV. THEOREM.

507. *The volume of any prism is equal to the product of its base and altitude.*



Any prism may be divided into triangular prisms by passing planes through one of the lateral edges and the corresponding diagonals of the base.

The volume of each triangular prism is equal to the product of its base and altitude (§ 506).

Hence, the sum of the volumes of the triangular prisms is equal to the sum of their bases multiplied by their common altitude.

Therefore, the volume of the given prism is equal to the product of its base and altitude.

508. COR. I. *Two prisms having equivalent bases and equal altitudes are equivalent.*

509. COR. II. 1. *Two prisms having equal altitudes are to each other as their bases.*

2. *Two prisms having equivalent bases are to each other as their altitudes.*

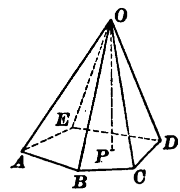
3. *Any two prisms are to each other as the products of their bases by their altitudes.*

Ex. 16. Find the lateral area and volume of a regular hexagonal prism, each side of whose base is 3, and whose altitude is 9.

PYRAMIDS.

DEFINITIONS.

510. A *pyramid* is a polyedron bounded by a polygon, and a series of triangles having a common vertex; as *O–ABCDE.*



The polygon is called the *base* of the pyramid, and the common vertex of the triangular faces is called the *vertex.*

The triangular faces are called the *lateral faces,* and their intersections the *lateral edges.*

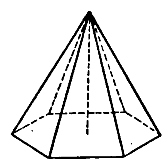
The sum of the areas of the lateral faces is called the *lateral area.*

511. The *altitude* of a pyramid is the perpendicular distance from the vertex to the plane of the base.

512. A pyramid is called *triangular, quadrangular,* etc., according as its base is a triangle, quadrilateral, etc.

NOTE. A triangular pyramid is a tetraedron (§ 467).

513. A *regular pyramid* is a pyramid whose base is a regular polygon, and whose vertex lies in the perpendicular erected at the centre of the base.



514. The lateral edges of a regular pyramid are equal. (§ 407, I.)

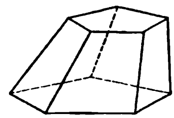
Whence, the lateral faces are equal isosceles triangles. (§ 69.)

515. The *slant height* of a regular pyramid is the perpendicular distance from the vertex of the pyramid to any side of the base.

Or, it is the straight line drawn from the vertex of the pyramid to the middle point of any side of the base. (§ 92.)

516. A *truncated pyramid* is that portion of a pyramid included between the base, and a plane cutting all the lateral edges.

517. A *frustum of a pyramid* is a truncated pyramid whose bases are parallel.



The *altitude* of the frustum is the perpendicular distance between the planes of its bases.

The lateral faces of a frustum of a pyramid are trapezoids. (§ 417.)

518. The lateral faces of a frustum of a regular pyramid are equal.

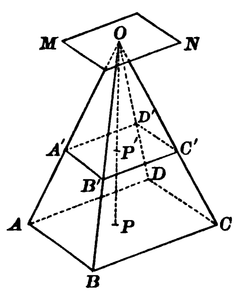
The *slant height* of a frustum of a regular pyramid is the altitude of any lateral face.

PROPOSITION XV. THEOREM.

519. *If a pyramid be cut by a plane parallel to its base,*

I. *The lateral edges and the altitude are divided proportionally.*

II. *The section is similar to the base.*



Let $A'C'$ be a plane parallel to the base of the pyramid $O\text{–}ABCD$, cutting the faces OAB, OBC, OCD, and ODA in the lines $A'B'$, $B'C'$, $C'D'$, and $D'A'$, and the altitude OP at P'.

I. To prove $\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC}$ etc. $= \frac{OP'}{OP}$.

Through O pass the plane MN parallel to $ABCD$.

Then, $$\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} \text{ etc.} = \frac{OP'}{OP}. \qquad (\S\ 425.)$$

II. To prove the section $A'B'C'D'$ similar to $ABCD$.

We have $A'B'$ parallel to AB, $B'C'$ to BC, etc. (§ 417.)

Then, $$\angle A'B'C' = \angle ABC,$$
$$\angle B'C'D' = \angle BCD, \text{ etc.} \qquad (\S\ 424.)$$

That is, the polygons $A'B'C'D'$ and $ABCD$ are mutually equiangular.

Again, the triangles $OA'B'$ and OAB are similar. (§ 258.)

Whence, $$\frac{OA'}{OA} = \frac{A'B'}{AB}. \qquad (1)$$

In like manner, $$\frac{OB'}{OB} = \frac{B'C'}{BC}, \text{ etc.}$$

But, $$\frac{OA'}{OA} = \frac{OB'}{OB}, \text{ etc.} \qquad (\S\ 519, \text{I.})$$

Whence, $$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD}, \text{ etc.}$$

That is, the polygons $A'B'C'D'$ and $ABCD$ have their homologous sides proportional.

Therefore, $A'B'C'D'$ and $ABCD$ are similar. (§ 252.)

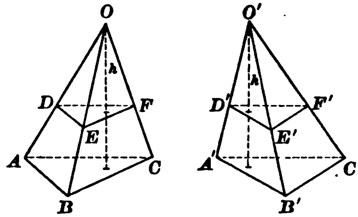
520. Cor. I. We have
$$\frac{\text{area } A'B'C'D'}{\text{area } ABCD} = \frac{\overline{A'B'}^2}{\overline{AB}^2}. \qquad (\S\ 323.)$$

But, from equation (1) of § 519,
$$\frac{A'B'}{AB} = \frac{OA'}{OA} = \frac{OP'}{OP}. \qquad (\S\ 519, \text{I.})$$

Whence, $$\frac{\text{area } A'B'C'D'}{\text{area } ABCD} = \frac{\overline{OP'}^2}{\overline{OP}^2}.$$

That is, *the areas of two parallel sections of a pyramid are to each other as the squares of their distances from the vertex.*

521. Cor. II. *If two pyramids have equal altitudes and equivalent bases, sections parallel to their bases equally distant from their vertices are equivalent.*



Let the bases of the pyramids $O\text{–}ABC$ and $O'\text{–}A'B'C'$ be equivalent, and let the altitude of each pyramid be H.

Let DEF and $D'E'F'$ be sections parallel to the bases, at the distance h from the vertices.

To prove DEF equivalent to $D'E'F'$.

Now, $\dfrac{\text{area } DEF}{\text{area } ABC} = \dfrac{h^2}{H^2}$, and $\dfrac{\text{area } D'E'F'}{\text{area } A'B'C'} = \dfrac{h^2}{H^2}$. (§ 520.)

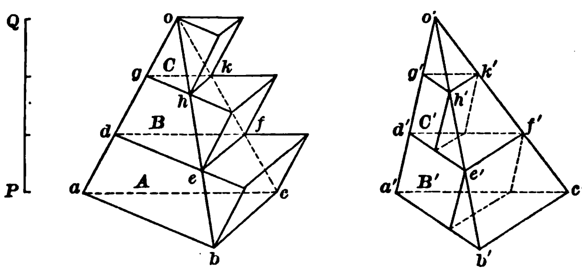
Whence, $$\frac{\text{area } DEF}{\text{area } ABC} = \frac{\text{area } D'E'F'}{\text{area } A'B'C'}.$$

But by hypothesis, area ABC = area $A'B'C'$.

Therefore; area DEF = area $D'E'F'$.

Proposition XVI. Theorem.

522. *Two triangular pyramids having equal altitudes and equivalent bases are equivalent.*



Let $o\text{–}abc$ and $o'\text{–}a'b'c'$ be two triangular pyramids having equal altitudes and equivalent bases.

To prove vol. $o\text{–}abc$ = vol. $o'\text{–}a'b'c'$.

Place the pyramids with their bases in the same plane, and let PQ be their common altitude.

Divide PQ into any number of equal parts; and through the points of division pass planes parallel to the plane of the bases, cutting $o\text{–}abc$ in the sections def and ghk, and $o'\text{–}a'b'c'$ in the sections $d'e'f'$ and $g'h'k'$.

Then def is equivalent to $d'e'f'$, and ghk to $g'h'k'$. (§ 521.)

With abc, def, and ghk as *lower* bases, construct the prisms A, B, and C, having their lateral edges equal and parallel to ad; and with $d'e'f'$ and $g'h'k'$ as *upper* bases, construct the prisms B' and C', having their lateral edges equal and parallel to $a'd'$.

Then, the prism B is equivalent to B'. (§ 508.)

In like manner, C is equivalent to C'.

Hence, the sum of the prisms circumscribed about $o\text{–}abc$ exceeds the sum of the prisms inscribed in $o'\text{–}a'b'c'$ by the prism A.

But $o\text{–}abc$ is evidently less than the sum of the prisms A, B, and C; and it is greater than the sum of the inscribed prisms, equivalent to B' and C', which can be constructed with def and ghk as *upper* bases.

Again, $o'\text{–}a'b'c'$ is greater than the sum of the prisms B' and C'; and it is less than the sum of the circumscribed prisms, equivalent to A, B, and C, which can be constructed with $a'b'c'$, $d'e'f'$, and $g'h'k'$ as *lower* bases.

Hence, the difference of the volumes of the pyramids must be less than the difference of the volumes of the two systems of prisms, and must therefore be less than the volume of the prism A.

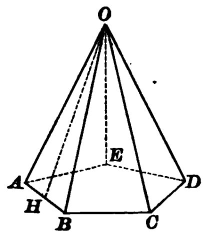
Now by sufficiently increasing the number of subdivisions of PQ, the volume of the prism A may be made less than any assigned volume, however small.

Therefore, the volumes of the pyramids cannot differ by any volume, however small.

Whence, vol. $o\text{–}abc$ = vol. $o'\text{–}a'b'c'$.

Proposition XVII. Theorem.

523. *The lateral area of a regular pyramid is equal to the perimeter of its base multiplied by one-half its slant height.*



Let OH be the slant height of the regular pyramid $O\text{–}ABCDE$.

To prove

$$\text{lat. area } O\text{–}ABCDE = (AB + BC + \text{etc.}) \times \tfrac{1}{2}\, OH.$$

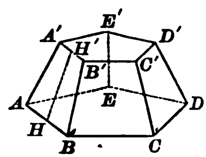
Now, area $OAB = AB \times \frac{1}{2}\, OH$. (§ 313.)

Also, area $OBC = BC \times \frac{1}{2}\, OH$; etc. (§ 515.)

Adding these equations, we have

$$\text{lat. area } O\text{–}ABCDE = (AB + BC + \text{etc.}) \times \tfrac{1}{2}\, OH.$$

524. Cor. *The lateral area of a frustum of a regular pyramid is equal to one-half the sum of the perimeters of its bases, multiplied by its slant height.*



Let HH' be the slant height of the frustum of a regular pyramid AD'.

To prove

$$\text{lat. area } AD' = \tfrac{1}{2}(AB + A'B' + BC + B'C' + \text{etc.}) \times HH'.$$

Now, area $AA'B'B = \frac{1}{2}(AB + A'B') \times HH'$. (§ 317.)

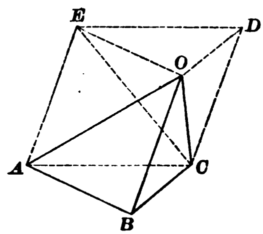
Also, area $BB'C'C = \frac{1}{2}(BC + B'C') \times HH'$; etc.

Adding these equations, we have

$$\text{lat. area } AD' = \tfrac{1}{2}(AB + A'B' + BC + B'C' + \text{etc.}) \times HH'.$$

Proposition XVIII. Theorem.

525. *A triangular pyramid is equivalent to one-third of a triangular prism having the same base and altitude.*



Let $O\text{–}ABC$ be a triangular pyramid.

Upon the base ABC, construct the prism $ABC\text{–}ODE$, having its lateral edges equal and parallel to OB.

To prove vol. $O\text{–}ABC = \frac{1}{3}$ vol. $ABC\text{–}ODE$.

The prism $ABC\text{–}ODE$ is composed of the triangular pyramid $O\text{–}ABC$, and the quadrangular pyramid $O\text{–}ACDE$.

Divide $O\text{–}ACDE$ into two triangular pyramids, $O\text{–}ACE$ and $O\text{–}CDE$, by passing a plane through O, C, and E.

Now, $O\text{–}ACE$ and $O\text{–}CDE$ have the same altitude.

And since CE is a diagonal of the parallelogram $ACDE$ they have equal bases, ACE and CDE. (§ 106.)

Hence, vol. $O\text{–}ACE =$ vol. $O\text{–}CDE$. (§ 522.)

Again, the pyramid $O\text{–}CDE$ may be regarded as having its vertex at C, and the triangle ODE for its base.

Then, the pyramids $O\text{–}ABC$ and $C\text{–}ODE$ have the same altitude. (§ 422.)

They have also equal bases, ABC and ODE. (§ 472.)

Hence, vol. $O\text{–}ABC =$ vol. $C\text{–}ODE$. (§ 522.)

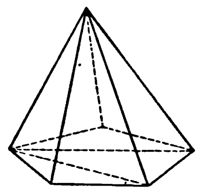
Then, vol. $O\text{–}ABC =$ vol. $O\text{–}ACE =$ vol. $O\text{–}CDE$.

Whence, vol. $O\text{–}ABC = \frac{1}{3}$ vol. $ABC\text{–}ODE$.

526. Cor. *The volume of a triangular pyramid is equal to one-third the product of its base and altitude.* (§ 506.)

PROPOSITION XIX. THEOREM.

527. *The volume of any pyramid is equal to one-third the product of its base and altitude.*



Any pyramid may be divided into triangular pyramids by passing planes through one of the lateral edges and the corresponding diagonals of the base.

The volume of each triangular pyramid is equal to one-third the product of its base and altitude (§ 526).

Hence, the sum of the volumes of the triangular pyramids is equal to the sum of their bases multiplied by one-third their common altitude.

Therefore, the volume of the given pyramid is equal to one-third the product of its base and altitude.

528. COR. I. *Two pyramids having equivalent bases and equal altitudes are equivalent.*

529. COR. II. 1. *Two pyramids having equal altitudes are to each other as their bases.*

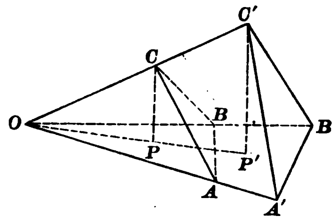
2. *Two pyramids having equivalent bases are to each other as their altitudes.*

3. *Any two pyramids are to each other as the products of their bases by their altitudes.*

530. SCH. The volume of any polyedron may be obtained by dividing it into pyramids.

Proposition XX. Theorem.

531. *Two tetraedrons having a triedral of one equal to a triedral of the other, are to each other as the products of the edges including the equal triedrals.*



Let V and V' denote the volumes of the tetraedrons O–ABC and O–$A'B'C'$, having the common triedral O.

To prove $$\frac{V}{V'} = \frac{OA \times OB \times OC}{OA' \times OB' \times OC'}.$$

Draw CP and $C'P'$ perpendicular to the face $OA'B'$.

Let their plane intersect $OA'B'$ in the line OPP'.

Now, OAB and $OA'B'$ are the bases, and CP and $C'P'$ the altitudes, of the triangular pyramids C–OAB and C'–$OA'B'$.

Whence, $$\frac{V}{V'} = \frac{OAB \times CP}{OA'B' \times C'P'} \qquad (\S\ 529, 3.)$$

$$= \frac{OAB}{OA'B'} \times \frac{CP}{C'P'}. \qquad (1)$$

But, $$\frac{OAB}{OA'B'} = \frac{OA \times OB}{OA' \times OB'}. \qquad (\S\ 322.)$$

Also, the right triangles OCP and $OC'P'$ are similar. (§ 257.)

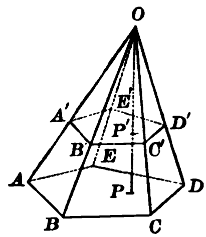
Whence, $$\frac{CP}{C'P'} = \frac{OC}{OC'}.$$

Substituting these values in (1), we have

$$\frac{V}{V'} = \frac{OA \times OB}{OA' \times OB'} \times \frac{OC}{OC'} = \frac{OA \times OB \times OC}{OA' \times OB' \times OC'}.$$

PROPOSITION XXI. THEOREM.

532. ***The volume of a frustum of a pyramid is equal to the sum of its bases and a mean proportional between its bases, multiplied by one-third its altitude.***



Let AD' be a frustum of any pyramid $O\text{–}ABCDE$.

Denote the area of the lower base by B, the area of the upper base by b, and the altitude by H.

To prove vol. $AD' = (B + b + \sqrt{B \times b}) \times \frac{1}{3} H$. (§ 232.)

Draw the altitude OP, cutting $A'B'C'D'E'$ at P'.

Now, vol. $AD' =$ vol. $O\text{–}ABCDE$ — vol. $O\text{–}A'B'C'D'E'$

$$= B \times \tfrac{1}{3}\, OP - b \times \tfrac{1}{3}\, OP' \qquad (\S\ 527.)$$

$$= B \times \tfrac{1}{3}\,(H + OP') - b \times \tfrac{1}{3}\, OP'$$

$$= B \times \tfrac{1}{3}\, H + B \times \tfrac{1}{3}\, OP' - b \times \tfrac{1}{3}\, OP'$$

$$= B \times \tfrac{1}{3}\, H + (B - b) \times \tfrac{1}{3}\, OP'. \qquad (1)$$

But, $$B : b = \overline{OP}^2 : \overline{OP'}^2. \qquad (\S\ 520.)$$

Taking the square root of each term, we have

$$\sqrt{B} : \sqrt{b} = OP : OP'. \qquad (\S\ 242.)$$

Then, $$\sqrt{B} - \sqrt{b} : \sqrt{b} = OP - OP' \text{ or } H : OP'. \qquad (\S\ 237.)$$

Whence, $$(\sqrt{B} - \sqrt{b}) \times OP' = \sqrt{b} \times H. \qquad (\S\ 231.)$$

Multiplying both members by $\sqrt{B} + \sqrt{b}$,

$$(B - b) \times OP' = (\sqrt{B \times b} + b) \times H.$$

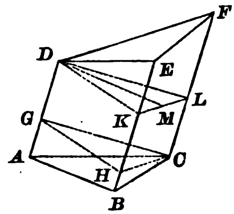
Substituting in (1), we have

$$\text{vol. } AD' = B \times \tfrac{1}{3}\, H + (\sqrt{B \times b} + b) \times \tfrac{1}{3}\, H$$

$$= (B + b + \sqrt{B \times b}) \times \tfrac{1}{3}\, H.$$

Proposition XXII. Theorem.

533. *The volume of a truncated triangular prism is equal to the product of a right section by one-third the sum of its lateral edges.*



Let GHC and DKL be right sections of the truncated triangular prism $ABC\text{-}DEF$.

To prove

$$\text{vol. } ABC\text{-}DEF = GHC \times \tfrac{1}{3}(AD + BE + CF).$$

Draw DM perpendicular to KL.

The truncated prism is composed of the triangular prism $GHC\text{-}DKL$, and the pyramids $D\text{-}EKLF$ and $C\text{-}ABHG$.

Now since the lateral edges of a prism are equal,

vol. $GHC\text{-}DKL = GHC \times GD$ (§ 506.)

$= GHC \times \frac{1}{3}(GD + HK + CL)$. (1)

Again, DM is the altitude of the pyramid $D\text{-}EKLF$. (§ 440.)

Whence, vol. $D\text{-}EKLF = EKLF \times \frac{1}{3} DM$ (§ 527.)

$= \frac{1}{2}(KE + LF) \times KL \times \frac{1}{3} DM$ (§ 317.)

$= \frac{1}{2} KL \times DM \times \frac{1}{3}(KE + LF)$.

But, $\frac{1}{2} KL \times DM = \text{area } DKL = \text{area } GHC$. (§ 313.)

Hence, vol. $D\text{-}EKLF = GHC \times \frac{1}{3}(KE + LF)$. (2)

In like manner, we may prove

vol. $C\text{-}ABHG = GHC \times \frac{1}{3}(AG + BH)$. (3)

Adding (1), (2), and (3), we have

vol. $ABC\text{-}DEF$

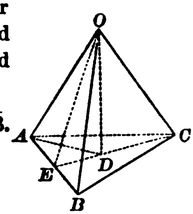
$$= GHC \times \tfrac{1}{3}(\overline{AG + GD} + \overline{BH + HK + KE} + \overline{CL + LF})$$

$$= GHC \times \tfrac{1}{3}(AD + BE + CF).$$

534. Cor. *The volume of a truncated right triangular prism is equal to the product of its base by one-third the sum of its lateral edges.*

EXERCISES.

17. Each side of the base of a regular triangular pyramid is 6, and its altitude is 4. Find its lateral edge, lateral area, and volume.



Let D be the centre of the base of the regular triangular pyramid $O\text{–}ABC$, and draw OD and AD; also, draw CDE perpendicular to AB, and join OE.

Now, $AD = AB \div \sqrt{3}$ (§ 357) $= \dfrac{6}{\sqrt{3}} = 2\sqrt{3}$.

Then, lat. edge OA

$= \sqrt{\overline{OD}^2 + \overline{AD}^2} = \sqrt{16 + 12} = \sqrt{28} = 2\sqrt{7}$.

The slant ht. $OE = \sqrt{\overline{OA}^2 - \overline{AE}^2} = \sqrt{28 - 9} = \sqrt{19}$.

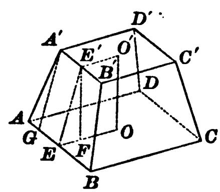
Then, lat. area $O\text{–}ABC = 9\sqrt{19}$ (§ 523).

Again, $CE = \sqrt{\overline{BC}^2 - \overline{BE}^2} = \sqrt{36 - 9} = \sqrt{27} = 3\sqrt{3}$.

Then, area $ABC = \frac{1}{2} \times 6 \times 3\sqrt{3}$ (§ 313) $= 9\sqrt{3}$.

Whence, vol. $O\text{–}ABC = \frac{1}{3} \times 9\sqrt{3} \times 4$ (§ 526) $= 12\sqrt{3}$.

18. Find the lateral edge, lateral area, and volume of a frustum of a regular quadrangular pyramid, the sides of whose bases are 17 and 7, and whose altitude is 12.



Let O and O' be the centres of the bases of the frustum of a regular quadrangular pyramid AC'.

Draw OE and $O'E'$ perpendicular to AB and $A'B'$, and $E'F$ and $A'G$ perpendicular to OE and AB; also, draw OO' and EE'.

Now, $EF = OE - O'E' = 8\frac{1}{2} - 3\frac{1}{2} = 5$.

Then, slant ht. $EE' = \sqrt{\overline{EF}^2 + \overline{E'F}^2} = \sqrt{25 + 144} = \sqrt{169} = 13$.

Whence, lat. area $AC' = \frac{1}{2}(68 + 28) \times 13$ (§ 524) $= 624$.

Again, $AG = AE - A'E' = 8\frac{1}{2} - 3\frac{1}{2} = 5$; and $A'G = EE' = 13$.

Whence, lat. edge $AA' = \sqrt{\overline{AG}^2 + \overline{A'G}^2} = \sqrt{25 + 169} = \sqrt{194}$.

Again, area $AC = 17^2 = 289$, and area $A'C' = 7^2 = 49$.

Then, a mean proportional between the areas of the bases

$$= \sqrt{17^2 \times 7^2} = 17 \times 7 = 119.$$

Whence, vol. $AC' = (289 + 49 + 119) \times 4$ (§ 532) $= 1828$.

Find the lateral edge, lateral area, and volume

19. Of a regular triangular pyramid, each side of whose base is 12, and whose altitude is 15.

20. Of a regular quadrangular pyramid, each side of whose base is 3, and whose altitude is 5.

21. Of a regular hexagonal pyramid, each side of whose base is 4, and whose altitude is 9.

22. Of a frustum of a regular triangular pyramid, the sides of whose bases are 18 and 6, and whose altitude is 24.

23. Of a frustum of a regular quadrangular pyramid, the sides of whose bases are 9 and 5, and whose altitude is 10.

24. Of a frustum of a regular hexagonal pyramid, the sides of whose bases are 8 and 4, and whose altitude is 12.

25. Find the volume of a truncated right triangular prism, the sides of whose base are 5, 12, and 13, and whose lateral edges are 3, 7, and 5, respectively.

26. Find the volume of a truncated regular quadrangular prism, each side of whose base is 8, and whose lateral edges, taken in order, are 2, 6, 8, and 4, respectively.

27. The volume of a cube is $4\frac{17}{27}$ cu. ft. What is the area of its entire surface in square inches?

28. A box made of 2 in. plank, without a cover, measures on the outside 3 ft. 2 in. long, 2 ft. 3 in. wide, and 1 ft. 6 in. deep. How many cubic feet of material were used in its construction?

29. The volume of a right prism is 2310, and its base is a right triangle whose legs are 20 and 21. Find its lateral area.

30. Find the lateral area and volume of a right triangular prism, the sides of whose base are 4, 7, and 9, and whose altitude is 8.

31. Find the volume of a truncated right triangular prism, whose lateral edges are 11, 14, and 17, having for its base an isosceles triangle whose sides are 10, 13, and 13.

32. The altitude of a pyramid is 12 in., and its base is a square 9 in. on a side. What is the area of a section parallel to the base, whose distance from the vertex is 8 in.?

33. The altitude of a pyramid is 20 in., and its base is a rectangle whose dimensions are 10 in. and 15 in. What is the distance from the vertex of a section parallel to the base, whose area is 54 sq. in.?

34. The diagonal of a cube is $8\sqrt{3}$. Find its volume, and the area of its entire surface.

35. A trench is 124 ft. long, $2\frac{1}{2}$ ft. deep, 6 ft. wide at the top, and 5 ft. wide at the bottom. How many cubic feet of water will it contain?

36. The volume of a regular triangular prism is $96\sqrt{3}$, and one side of its base is 8. Find its lateral area.

37. The lateral area and volume of a regular hexagonal prism are 60 and $15\sqrt{3}$, respectively. Find its altitude, and one side of its base.

38. The slant height and lateral edge of a regular quadrangular pyramid are 25 and $\sqrt{674}$, respectively. Find its lateral area and volume.

39. The altitude and slant height of a regular hexagonal pyramid are 15 and 17, respectively. Find its lateral edge and volume.

40. The lateral edge of a frustum of a regular hexagonal pyramid is 10, and the sides of its bases are 10 and 4, respectively. Find its lateral area and volume.

41. Find the lateral area and volume of a regular quadrangular pyramid, the area of whose base is 100, and whose lateral edge is 13.

42. Find the lateral area and volume of a frustum of a regular triangular pyramid, the sides of whose bases are 12 and 6, and whose lateral edge is 5.

43. The lateral edges of a frustum of a quadrangular pyramid are equal; and its bases are rectangles, whose sides are 27 and 15, and 9 and 5, respectively. If the altitude of the frustum is 12, find its lateral area and volume.

44. Any straight line drawn through the centre of a parallelopiped, terminating in a pair of opposite faces, is bisected at that point.

45. The lateral surface of a pyramid is greater than its base.

46. The volume of a regular prism is equal to its lateral area, multiplied by one-half the apothem of its base.

47. The volume of a regular pyramid is equal to its lateral area, multiplied by one-third the distance from the centre of its base to any lateral face.

48. If E, F, G, and H are the middle points of the edges AB, AD, CD, and BC, respectively, of the tetraedron $ABCD$, prove that $EFGH$ is a parallelogram.

49. Find the area of the base of a regular quadrangular pyramid, whose lateral faces are equilateral triangles, and whose altitude is 5.

50. Two tetraedrons are equal if two faces and the included diedral of one are equal, respectively, to two faces and the included diedral of the other, if the equal parts are similarly placed.

51. Two tetraedrons are equal if three faces of one are equal, respectively, to three faces of the other, if the equal parts are similarly placed.

52. Find the area of the entire surface and the volume of a triangular pyramid, each of whose edges is 2.

53. The areas of the bases of a frustum of a pyramid are 12 and 75, and its altitude is 9. What is the altitude of the pyramid?

54. The sum of two opposite lateral edges of a truncated parallelopiped is equal to the sum of the other two lateral edges.

55. The volume of a truncated parallelopiped is equal to the area of a right section, multiplied by one-fourth the sum of the lateral edges.

56. A plane passed through the centre of a parallelopiped divided it into two equivalent solids.

57. If $ABCD$ is a rectangle, and EF any straight line not in its plane parallel to AB, the volume of the solid bounded by the figures $ABCD$, $ABFE$, $CDEF$, ADE, and BCF, is equal to

$$\tfrac{1}{6}\, h \times AD \times (2\, AB + EF),$$

where h is the perpendicular from E to $ABCD$. (§ 533.)

58. If $ABCD$ and $EFGH$ are rectangles lying in parallel planes, AB and BC being parallel to EF and FG, respectively, the solid bounded by the figures $ABCD$, $EFGH$, $ABFE$, $BCGF$, $CDHG$, and $DAEH$, is called a *rectangular prismoid.*

$ABCD$ and $EFGH$ are called the *bases* of the rectangular prismoid, and the perpendicular distance between them the *altitude.*

Prove that the volume of a rectangular prismoid is equal to the sum of its bases, plus four times a section equidistant from the bases, multiplied by one-sixth the altitude. (Ex. 57.)

59. Find the volume of a rectangular prismoid, the sides of whose bases are 10 and 7, and 6 and 5, respectively, and whose altitude is 9.

60. The volume of a triangular prism is equal to a lateral face, multiplied by one-half its perpendicular distance from any point in the opposite lateral edge.

61. The volume of a truncated right parallelopiped is equal to the area of its lower base, multiplied by the perpendicular drawn to the lower base from the centre of the upper base.

62. The perpendicular drawn to the lower base of a truncated right triangular prism from the intersection of the medians of the upper base, is equal to one-third the sum of the lateral edges.

63. The three planes passing through the lateral edges of a triangular pyramid, bisecting the sides of the base, meet in a common straight line.

64. A frustum of any pyramid is equivalent to the sum of three pyramids, having for their common altitude the altitude of the frustum, and for their bases the lower base, the upper base, and a mean proportional between the bases, of the frustum.

65. A monument is in the form of a frustum of a regular quadrangular pyramid 8 ft. in height, the sides of whose bases are 3 ft. and 2 ft., respectively, surmounted by a regular quadrangular pyramid 2 ft. in height. What is its weight, at 180 lb. to the cubic foot?

66. The altitude and lateral edge of a frustum of a regular triangular pyramid are 8 and 10, respectively, and each side of its upper base is $2\sqrt{3}$. Find its volume and lateral area.

67. A railway embankment, 1620 ft. in length, is $8\frac{1}{2}$ ft. wide at the top, $21\frac{1}{2}$ ft. wide at the bottom, and 6 ft. 4 in. high. How many cubic yards of earthwork does it contain?

68. The sides of the base, AB, BC, and CA, of a truncated right triangular prism ABC–DEF, are 15, 4, and 12, respectively, and the lateral edges, AD, BE, and CF, are 15, 7, and 10, respectively. Find the area of the upper base, DEF.

69. If $ABCD$ is a tetraedron, the section made by a plane parallel to each of the edges AB and CD is a parallelogram. (§ 415.)

70. In a tetraedron $ABCD$, a plane is drawn through the edge CD perpendicular to AB, intersecting the faces ABC and ABD in CE and ED. If the bisector of the angle CED meets CD at F, prove

$$CF : DF = \text{area } ABC : \text{area } ABD.$$

71. The sum of the squares of the four diagonals of any parallelopiped is equal to the sum of the squares of its twelve edges. (Ex. 75, p. 226.)

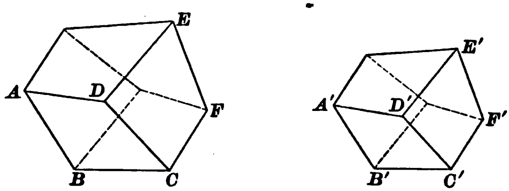
72. If the four diagonals of a quadrangular prism pass through a common point, the prism is a parallelopiped.

SIMILAR POLYEDRONS.

535. Def. Two polyedrons are said to be *similar* when they are bounded by the same number of faces, similar each to each and similarly placed, and have their homologous polyedrals equal.

Proposition XXIII. Theorem.

536. *The ratio of any two homologous edges of two similar polyedrons is equal to the ratio of any other two homologous edges.*



In the similar polyedrons AF and $A'F'$, let the edges AB and EF be homologous to the edges $A'B'$ and $E'F'$.

To prove $$\frac{AB}{A'B'} = \frac{EF}{E'F'}.$$

The face AC is similar to $A'C'$, and DF to $D'F'$. (§ 535.)

Whence, $$\frac{AB}{A'B'} = \frac{CD}{C'D'} = \frac{EF}{E'F'}. \qquad \text{(§ 253, I.)}$$

537. Cor. I. We have
$$\frac{\text{area } ABCD}{\text{area } A'B'C'D'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} \qquad \text{(§ 323.)}$$
$$= \frac{\overline{EF}^2}{\overline{E'F'}^2}. \qquad \text{(§ 536.)}$$

Therefore, *any two homologous faces of two similar polyedrons are to each other as the squares of any two homologous edges.*

538. Cor. II. We have

$$\frac{ABCD}{A'B'C'D'} = \frac{CDEF}{C'D'E'F'} = \text{etc.} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad (\S\ 537.)$$

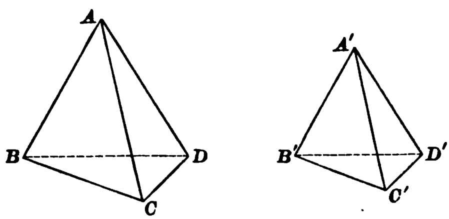
Whence,

$$\frac{ABCD + CDEF + \text{etc.}}{A'B'C'D' + C'D'E'F' + \text{etc.}} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad (\S\ 239.)$$

Hence, *the entire surfaces of two similar polyedrons are to each other as the squares of any two homologous edges.*

Proposition XXIV. Theorem.

539. *Two tetraedrons are similar when the faces including a triedral of one are similar to the faces including a triedral of the other, and similarly placed.*



In the tetraedrons $ABCD$ and $A'B'C'D'$, let the faces ABC, ACD, and ADB be similar to the faces $A'B'C'$, $A'C'D'$, and $A'D'B'$, respectively.

To prove $ABCD$ and $A'B'C'D'$ similar.

From the given similar faces, we have

$$\frac{BC}{B'C'} = \frac{AC}{A'C'} = \frac{CD}{C'D'} = \frac{AD}{A'D'} = \frac{BD}{B'D'}.$$

Hence, the faces BCD and $B'C'D'$ are similar. (§ 259.)

Again, since the angles BAC, CAD, and DAB are equal respectively to the angles $B'A'C'$, $C'A'D'$, and $D'A'B'$, the triedrals $A\text{–}BCD$ and $A'\text{–}B'C'D'$ are equal. (§ 464, I.)

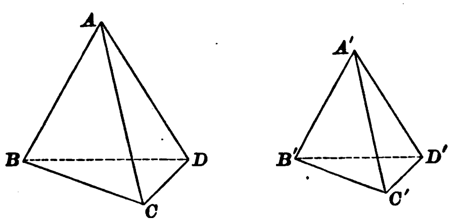
In like manner, any two homologous triedrals are equal.

Hence, $ABCD$ and $A'B'C'D'$ are similar. (§ 535.)

540. Cor. *If a tetraedron be cut by a plane parallel to one of its faces, the tetraedron cut off is similar to the given tetraedron.*

Proposition XXV. Theorem.

541. *Two tetraedrons are similar when a diedral of one is equal to a diedral of the other, and the faces including the equal diedrals are similar each to each, and similarly placed.*



In the tetraedrons $ABCD$ and $A'B'C'D'$, let the diedral AB be equal to $A'B'$; and let the faces ABC and ABD be similar to $A'B'C'$ and $A'B'D'$, respectively.

To prove $ABCD$ and $A'B'C'D'$ similar.

Let the tetraedron $A'B'C'D'$ be applied to $ABCD$, so that the diedral $A'B'$ shall coincide with its equal AB, the point A' falling at A.

Then since $\angle B'A'C' = \angle BAC$, and $\angle B'A'D' = \angle BAD$, the edge $A'C'$ will coincide with AC, and $A'D'$ with AD.

Therefore, $\angle C'A'D' = \angle CAD$.

Again, from the given similar faces, we have

$$\frac{A'C}{AC} = \frac{A'B'}{AB} = \frac{A'D'}{AD}.$$

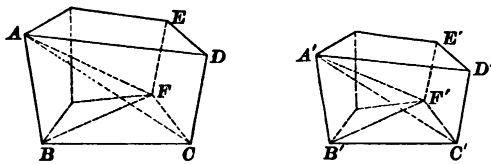
Hence, the triangle $C'A'D'$ is similar to CAD. (§ 260.)

Then, the faces including the triedral A'–$B'C'D'$ are similar to the faces including the triedral A–BCD, and similarly placed.

Therefore, $ABCD$ and $A'B'C'D'$ are similar. (§ 539.)

Proposition XXVI. Theorem.

542. *Two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.*



Let AF and $A'F'$ be two similar polyedrons, the vertices A and A' being homologous.

To prove that they may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.

Divide all the faces of AF, except those having A as a vertex, into triangles; and draw straight lines from A to their vertices.

In like manner, divide all the faces of $A'F'$, except those having A' as a vertex, into triangles similar to those in AF, and similarly placed. (§ 267.)

Draw straight lines from A' to their vertices.

The given polyedrons are then decomposed into the same number of tetraedrons, similarly placed.

Let $ABCF$ and $A'B'C'F'$ be two homologous tetraedrons.

The triangles ABC and BCF are similar to $A'B'C'$ and $B'C'F'$, respectively. (§ 267.)

And since the given polyedrons are similar, the homologous diedrals BC and $B'C'$ are equal.

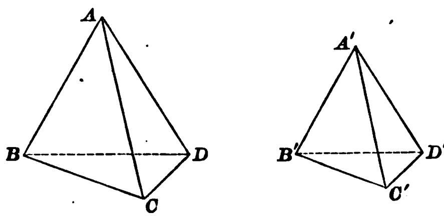
Therefore, $ABCF$ and $A'B'C'F'$ are similar. (§ 541.)

In like manner, we may prove any two homologous tetraedrons similar.

Hence, the given polyedrons are decomposed into the same number of tetraedrons, similar each to each, and similarly placed.

Proposition XXVII. Theorem.

543. *Two similar tetraedrons are to each other as the cubes of their homologous edges.*



Let V and V' denote the volumes of the similar tetraedrons $ABCD$ and $A'B'C'D'$, the vertices A and A' being homologous.

To prove $$\frac{V}{V'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

Since the triedrals at A and A' are equal, we have

$$\frac{V}{V'} = \frac{AB \times AC \times AD}{A'B' \times A'C' \times A'D'} \qquad (\S\ 531.)$$

$$= \frac{AB}{A'B'} \times \frac{AC}{A'C'} \times \frac{AD}{A'D'}. \qquad (1)$$

But, $$\frac{AC}{A'C'} = \frac{AD}{A'D'} = \frac{AB}{A'B'}. \qquad (\S\ 536.)$$

Substituting in (1), we have

$$\frac{V}{V'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

544. Cor. *Any two similar polyedrons are to each other as the cubes of their homologous edges.*

For any two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each (§ 542).

And any two homologous tetraedrons are to each other as the cubes of their homologous edges (§ 543), or as the cubes of any two homologous edges of the polyedrons (§ 536).

REGULAR POLYEDRONS.

545. Def. A *regular polyedron* is a polyedron whose faces are equal regular polygons, and whose polyedrals are all equal.

Proposition XXVIII. Theorem.

546. *Not more than five regular convex polyedrons are possible.*

A convex polyedral must have at least three faces, and the sum of its face angles must be less than 360° (§ 463).

1. *With equilateral triangles.*

Since the angle of an equilateral triangle is 60°, we may form a convex polyedral by combining either 3, 4, or 5 equilateral triangles.

Not more than 5 equilateral triangles can be combined to form a convex polyedral. (§ 463.)

Hence, not more than three regular convex polyedrons can be formed with equilateral triangles.

2. *With squares.*

Since the angle of a square is 90°, we may form a convex polyedral by combining 3 squares.

Not more than 3 squares can be combined to form a convex polyedral.

Hence, not more than one regular convex polyedron can be formed with squares.

3. *With regular pentagons.*

Since the angle of a regular pentagon is 108°, we may form a convex polyedral by combining 3 regular pentagons.

Not more than 3 regular pentagons can be combined to form a convex polyedral.

Hence, not more than one regular convex polyedron can be formed with regular pentagons.

Since the angle of a regular hexagon is 120°, no convex polyedral can be formed by combining regular hexagons.

In like manner, no convex polyedral can be formed by combining regular polygons of more than six sides.

Hence, not more than five regular convex polyedrons are possible.

Proposition XXIX. Problem.

547. *With a given edge, to construct a regular polyedron.*

We will now prove, by actual construction, that five regular convex polyedrons are possible:

1. The regular tetraedron, bounded by 4 equilateral triangles.

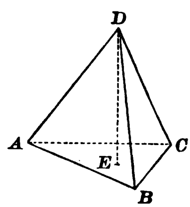
2. The regular hexaedron, or cube, bounded by 6 squares.

3. The regular octaedron, bounded by 8 equilateral triangles.

4. The regular dodecaedron, bounded by 12 regular pentagons.

5. The regular icosaedron, bounded by 20 equilateral triangles.

1. *To construct a regular tetraedron.*



Let AB be the given edge.

Construct the equilateral triangle ABC.

At its centre E, draw ED perpendicular to ABC; and take the point D so that $AD = AB$.

Draw AD, BD, and CD.

Then, $ABCD$ is a regular tetraedron.

For since A, B, and C are equally distant from E,

$$AD = BD = CD. \qquad (\S\ 407,\ \text{I.})$$

Hence, the six edges of the tetraedron are all equal.

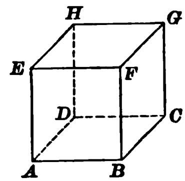
Then, the faces are equal equilateral triangles. (§ 69.)

And since the angles of the faces are all equal, the triedrals whose vertices are A, B, C, and D are equal.

(§ 464, I.)

Therefore, $ABCD$ is a regular tetraedron.

2. *To construct a regular hexaedron, or cube.*



Let AB be the given edge.

Construct the square $ABCD$.

Draw AE, BF, CG, and DH, each equal to AB and perpendicular to $ABCD$.

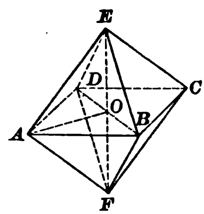
Draw EF, FG, GH, and HE.

Then, AG is a regular hexaedron.

For by construction, its faces are equal squares.

Hence, its triedrals are all equal. (§ 464, I.)

3. *To construct a regular octaedron.*



Let AB be the given edge.

Construct the square $ABCD$; through its centre O draw EOF perpendicular to $ABCD$, making $OE = OF = OA$.

Join the points E and F to A, B, C, and D.

Then, $AEFC$ is a regular octaedron.

For draw OA, OB, and OD.

Then in the right triangles AOB, AOE, and AOF,

$$OA = OB = OE = OF.$$

Therefore, $\triangle AOB = \triangle AOE = \triangle AOF$. (§ 63.)

Whence, $AB = AE = AF$. (§ 66.)

Then the eight edges terminating at E and F are all equal. (§ 407, I.)

Thus, the twelve edges of the octaedron are all equal, and the faces are equal equilateral triangles. (§ 69.)

Again, by construction, the diagonals of the quadrilateral $BEDF$ are equal, and bisect each other at right angles.

Hence, $BEDF$ is a square equal to $ABCD$, and OA is perpendicular to its plane. (§ 400.)

Therefore, the pyramids A–$BEDF$ and E–$ABCD$ are equal; and hence the polyedrals A–$BEDF$ and E–$ABCD$ are equal.

In like manner, any two polyedrals are equal.

Hence, $AEFC$ is a regular octaedron.

4. *To construct a regular dodecaedron.*

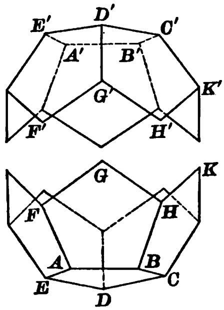


Fig. 1.

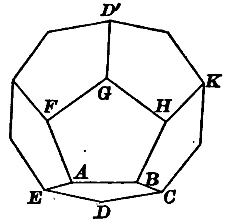


Fig. 2.

Let AB be the given edge.

Construct the regular pentagon $ABCDE$ (Fig. 1).

To $ABCDE$ join five equal regular pentagons, so inclined as to form equal triedrals at the vertices A, B, C, D, and E. (§ 464, I.)

Then there is formed a convex surface AK, composed of six regular pentagons, as shown in the lower portion of Fig. 1.

Construct a second surface $A'K'$ equal to AK, as shown in the upper portion of Fig. 1.

The surfaces AK and $A'K'$ may be combined as shown in Fig. 2, so as to form at F a triedral equal to that at A, having for its faces the regular pentagons about the vertices F and F' in Fig. 1. (§ 464, I.)

Then, AK is a regular dodecaedron.

For since G' falls at G, and the diedral FG and the face angles FGH and FGD' (Fig. 2) are equal respectively to the diedral and face angles of the triedral F, the faces about the vertex G will form a triedral equal to that at F.

Continuing in this way, it may be proved that at each of the vertices H, K, etc., there is formed a triedral equal to that at F.

Therefore, AK is a regular dodecaedron.

5. *To construct a regular icosaedron.*

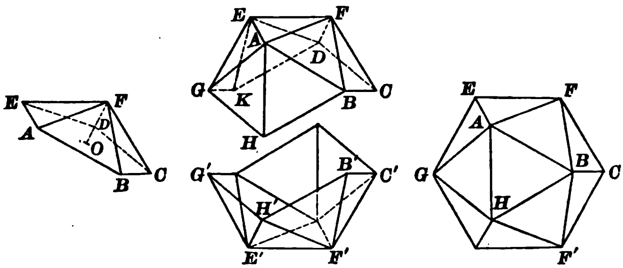


Fig. 1. *Fig. 2.* *Fig. 3.*

Let AB be the given edge.

Construct the regular pentagon $ABCDE$ (Fig. 1).

At its centre O draw OF perpendicular to $ABCDE$, making $AF = AB$.

Draw AF, BF, CF, DF, and EF.

Then $F\text{–}ABCDE$ is a regular pyramid whose lateral faces are equal equilateral triangles. (§ 69.)

Construct two other regular pyramids, $A\text{–}BFEGH$ and $E\text{–}AFDKG$, each equal to $F\text{–}ABCDE$.

Place them as shown in the upper portion of Fig. 2, so that the faces ABF and AEF of $A\text{–}BFEGH$, and the faces AEF and DEF of $E\text{–}AFDKG$, shall coincide with the corresponding faces of $F\text{–}ABCDE$.

Then there is formed a convex surface GC, composed of ten equilateral triangles.

Construct a second surface $G'C'$ equal to GC, as shown in the lower portion of Fig. 2.

Then the surfaces GC and $G'C'$ may be combined as shown in Fig. 3, so that the edges GH and HB shall coincide with $G'H'$ and $H'B'$, respectively.

For since the diedrals AH, $E'H'$, and $F'H'$ are equal to the diedrals of the polyedral F, the faces about the vertices H and H' may be made to form a polyedral at H equal to that at F. (§ 459.)

Then since the diedrals FB, AB, HB, and $F'B$ (Fig. 3) are equal to the diedrals of the polyedral F, the faces about the vertex B will form a polyedral equal to that at F.

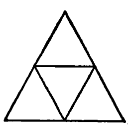
Continuing in this way, it may be shown that at each of the vertices C, D, etc., there is formed a polyedral equal to that at F.

Therefore, GC is a regular icosaedron.

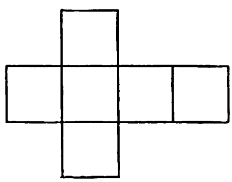
548. Sch. To construct the regular polyedrons, draw the following diagrams accurately on cardboard.

Cut the figures out entire, and on the interior lines cut the cardboard half through.

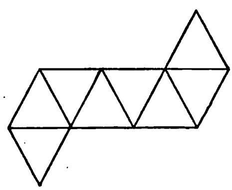
The edges may then be brought together so as to form the respective solids.



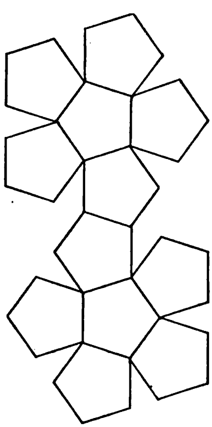
TETRAEDRON.



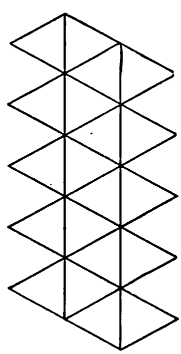
HEXAEDRON.



OCTAEDRON.



DODECAEDRON.



ICOSAEDRON.

EXERCISES.

73. If the volume of a pyramid whose altitude is 7 in. is 686 cu. in., what is the volume of a similar pyramid whose altitude is 12 in.?

74. If the volume of a prism whose altitude is 9 ft. is 171 cu. ft., what is the altitude of a similar prism whose volume is $50\frac{2}{3}$ cu. ft.?

75. Two bins of similar form contain, respectively, 375 and 648 bushels of wheat. If the first bin is 3 ft. 9 in. long, what is the length of the second?

76. A pyramid whose altitude is 10 in., weighs 24 lb. At what distance from its vertex must it be cut by a plane parallel to its base so that the frustum cut off may weigh 12 lb.?

77. An edge of a polyedron is 56, and the homologous edge of a similar polyedron is 21. The area of the entire surface of the second polyedron is 135, and its volume is 162. Find the area of the entire surface, and the volume, of the first polyedron.

78. The area of the entire surface of a tetraedron is 147, and its volume is 686. If the area of the entire surface of a similar tetraedron is 48, what is its volume?

79. The area of the entire surface of a tetraedron is 75, and its volume is 500. If the volume of a similar tetraedron is 32, what is the area of its entire surface?

80. The homologous edges of three similar tetraedrons are 3, 4, and 5, respectively. Find the homologous edge of a similar tetraedron equivalent to their sum.

81. State and prove the converse of Prop. XXVI.

82. The volume of a regular tetraedron is equal to the cube of its edge multiplied by $\frac{1}{12}\sqrt{2}$.

83. The volume of a regular octaedron is equal to the cube of its edge multiplied by $\frac{1}{3}\sqrt{2}$.

84. The volume of a regular tetraedron is $18\sqrt{2}$. Find the area of its entire surface.

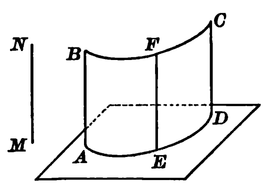
85. The sum of the perpendiculars drawn to the faces from any point within a regular tetraedron is equal to the altitude of the tetraedron.

BOOK VIII.

THE CYLINDER, CONE, AND SPHERE.

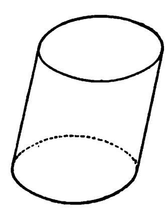
DEFINITIONS.

549. A *cylindrical surface* is a surface generated by a moving straight line, which constantly intersects a given curve, and in all of its positions is parallel to a given straight line, not in the plane of the curve.



Thus, if the line AB moves so as to constantly intersect the curve AD, and is constantly parallel to the line MN, not in the plane of the curve, it generates a cylindrical surface.

550. The moving straight line is called the *generatrix*, and the curve the *directrix*; any position of the generatrix, as EF, is called an *element* of the surface.



551. A *cylinder* is a solid bounded by a cylindrical surface, and two parallel planes.

The parallel planes are called the *bases* of the cylinder, and the cylindrical surface the *lateral surface.*

The *altitude* of a cylinder is the perpendicular distance between the planes of its bases.

NOTE. We shall use the phrase "element of a cylinder" to signify an *element of its lateral surface.*

552. It follows from the definition of § 551 that

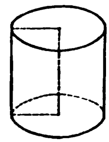
The elements of a cylinder are equal and parallel. (§ 418.)

553. A *right cylinder* is a cylinder whose elements are perpendicular to its bases.

A *circular cylinder* is a cylinder whose base is a circle.

The *axis* of a circular cylinder is a straight line drawn through the centre of its base parallel to its elements.

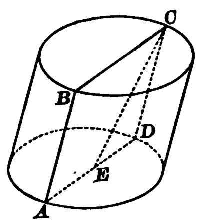
554. A right circular cylinder is called a *cylinder of revolution*; for it may be generated by the revolution of a rectangle about one of its sides as an axis.



555. *Similar* cylinders of revolution are cylinders generated by the revolution of similar rectangles about homologous sides as axes.

PROPOSITION I. THEOREM.

556. *A section of a cylinder made by a plane passing through an element is a parallelogram.*



Let $ABCD$ be a section of the cylinder AC, made by a plane passing through the element AB.

To prove $ABCD$ a parallelogram.

Draw CE in the plane $ABCD$ parallel to AB.

Then CE is an element of the lateral surface. (§ 552.)

Therefore, CE must be the intersection of the plane $ABCD$ with the lateral surface of the cylinder.

Hence, CE coincides with CD, and CD is parallel to AB.

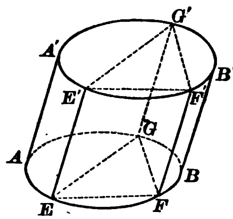
Again, AD is parallel to BC. (§ 417.)

Therefore, $ABCD$ is a parallelogram.

557. Cor. *A section of a right cylinder made by a plane passing through an element is a rectangle.*

Proposition II. Theorem.

558. *The bases of a cylinder are equal.*



Let AB' be a cylinder.

To prove its bases AB and $A'B'$ equal.

Let E', F', and G' be any three points in the perimeter of $A'B'$, and draw the elements EE', FF', and GG'.

Draw EF, FG, GE, $E'F'$, $F'G'$, and $G'E'$.

Now, EE' and FF' are equal and parallel. (§ 552.)

Therefore, $EE'F'F$ is a parallelogram. (§ 109.)

Hence, $E'F' = EF$. (§ 104.)

Similarly, $E'G' = EG$, and $F'G' = FG$.

Therefore, $\triangle E'F'G' = \triangle EFG$. (§ 69.)

Then the base $A'B'$ may be superposed upon AB so that the points E', F', and G' shall fall at E, F, and G.

But E' is any point in the perimeter of $A'B'$.

Hence, *every* point in the perimeter of $A'B'$ will fall in the perimeter of AB, and $A'B'$ is equal to AB.

559. Cor. I. *The sections of a cylinder made by two parallel planes cutting all its elements are equal.*

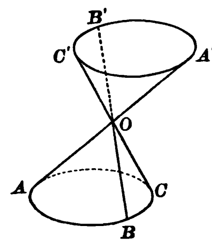
For they are the bases of a cylinder.

560. Cor. II. *A section of a cylinder made by a plane parallel to the base is equal to the base.*

THE CONE.

DEFINITIONS.

561. A *conical surface* is a surface generated by a moving straight line, which constantly intersects a given curve, and passes through a given point not in the plane of the curve.



Thus, if the line OA moves so as to constantly intersect the curve ABC, and constantly passes through the point O, not in the plane of the curve, it generates a conical surface.

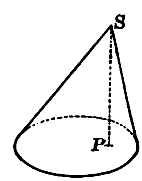
562. The moving straight line is called the *generatrix*, and the curve the *directrix*.

The given point is called the *vertex*; and any position of the generatrix, as OB, is called an *element* of the surface.

563. If the generatrix be supposed indefinite in length, it will generate two conical surfaces, $O\text{–}A'B'C'$ and $O\text{–}ABC$.

These are called the *upper* and *lower nappes*, respectively.

564. A *cone* is a solid bounded by a conical surface, and a plane cutting all its elements.



The plane is called the *base* of the cone, and the curved surface the *lateral surface.*

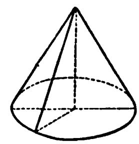
The *altitude* of a cone is the perpendicular distance from the vertex to the plane of the base.

565. A *circular cone* is a cone whose base is a circle.

The *axis* of a circular cone is a straight line drawn from the vertex to the centre of the base.

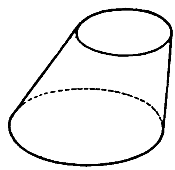
566. A *right circular cone* is a circular cone whose axis is perpendicular to its base.

567. A right circular cone is called a *cone of revolution,* for it may be generated by the revolution of a right triangle about one of its legs as an axis.



568. *Similar* cones of revolution are cones generated by the revolution of similar right triangles about homologous legs as axes.

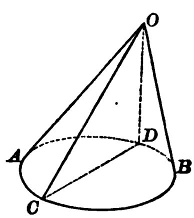
569. A *frustum of a cone* is that portion of a cone included between the base and a plane parallel to the base.



The *altitude* of a frustum is the perpendicular distance between the planes of its bases.

PROPOSITION III. THEOREM.

570. *A section of a cone made by a plane passing through the vertex is a triangle.*



Let OCD be a section of the cone OAB, made by a plane passing through the vertex O.

To prove OCD a triangle.

Draw straight lines in the plane OCD from O to the points C and D.

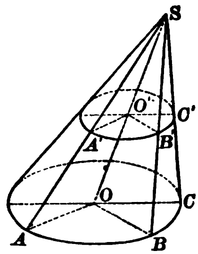
These lines are elements of the lateral surface. (§ 562.)

Then they must be the lines of intersection of the plane OCD with the lateral surface of the cone.

Therefore, OC and OD are straight lines, and OCD is a triangle.

Proposition IV. Theorem.

571. *A section of a circular cone made by a plane parallel to the base is a circle.*



Let $A'B'C'$ be a section of the circular cone $S\text{–}ABC$, made by a plane parallel to the base.

To prove $A'B'C'$ a circle.

Let the axis OS intersect the plane $A'B'C'$ at O'.

Let A' and B' be any two points in the perimeter $A'B'C'$.

Let the planes determined by these points and OS intersect the base in the radii OA and OB, and the lateral surface in the elements SA and SB. (§ 570.)

Then, $O'A'$ is parallel to OA, and $O'B'$ to OB. (§ 417.)

Therefore, the triangles $A'O'S$ and $B'O'S$ are similar to the triangles AOS and BOS. (§ 258.)

Whence, $$\frac{O'A'}{OA} = \frac{SO'}{SO}, \text{ and } \frac{O'B'}{OB} = \frac{SO'}{SO}.$$

Then, $$\frac{O'A'}{OA} = \frac{O'B'}{OB}.$$

But, $$OA = OB. \quad (\S\ 143.)$$

Whence, $$O'A' = O'B'.$$

Now A' and B' are *any* two points in the perimeter $A'B'C'$.

Therefore, the section $A'B'C'$ is a circle.

572. Cor. *The axis of a circular cone passes through the centre of every section parallel to the base.*

THE SPHERE.

Definitions.

573. A *sphere* is a solid bounded by a surface, all points of which are equally distant from a point within called the *centre.*

574. A *radius* of a sphere is a straight line drawn from the centre to the surface.

A *diameter* is a straight line drawn through the centre, having its extremities in the surface.

575. It follows from the definition of § 574 that

All radii of a sphere are equal.

Also, all its diameters are equal, since each is the sum of two radii.

576. A sphere may be generated by the revolution of a semicircle about its diameter as an axis.

577. *Two spheres are equal when their radii are equal.*

For they can evidently be applied one to the other so that their surfaces shall coincide throughout.

578. Conversely, *the radii of equal spheres are equal.*

579. A straight line or plane is said to be *tangent to a sphere* when it has but one point in common with the surface of the sphere.

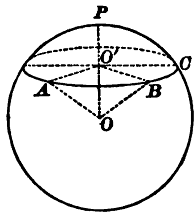
The common point is called the *point of contact,* or *point of tangency.*

580. A polyedron is said to be *inscribed in a sphere* when its vertices lie in the surface of the sphere; in this case the sphere is said to be circumscribed about the polyedron.

A polyedron is said to be *circumscribed about a sphere* when its faces are tangent to the sphere; in this case the sphere is said to be inscribed in the polyedron.

Proposition V. Theorem.

581. *A section of a sphere made by a plane is a circle.*



Let ABC be a section of the sphere APC made by a plane.

To prove ABC a circle.

Let O be the centre of the sphere.

Draw OO' perpendicular to the plane ABC.

Let A and B be any two points in the perimeter of ABC, and draw OA, OB, $O'A$, and $O'B$.

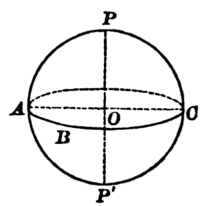
Now, $OA = OB$. (§ 575.)

Whence, $O'A = O'B$. (§ 408.)

But A and B are *any* two points in the perimeter of ABC.

Therefore, ABC is a circle.

582. Def. A *great circle* of a sphere is a section made by a plane passing through the centre; as ABC.



A *small circle* is a section made by a plane which does not pass through the centre.

The diameter perpendicular to a circle of a sphere is called the *axis* of the circle, and its extremities are called the *poles*.

583. Cor. I. *The axis of a circle of a sphere passes through the centre of the circle.*

584. Cor. II. *All great circles of a sphere are equal.*

For their radii are radii of the sphere.

585. Cor. III. *Every great circle bisects the sphere and its surface.*

For if the parts be separated, and placed so that their plane surfaces coincide, the spherical surfaces falling on the same side of this plane, the two spherical surfaces will coincide throughout; for all points of either are equally distant from the centre.

586. Cor. IV. *Any two great circles bisect each other.*

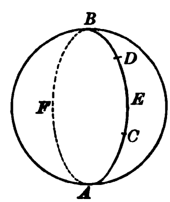
For the intersection of their planes passes through the centre of the sphere, and hence is a diameter of each circle.

587. Cor. V. *An arc of a great circle, less than a semi-circumference, may be drawn between any two points on the surface of a sphere, and but one.*

For the two points, together with the centre of the sphere, determine a plane which intersects the surface of the sphere in the arc required.

Note. If the points lie at the extremities of a diameter of the sphere, an indefinitely great number of arcs of great circles may be drawn between them; for an indefinitely great number of planes can be drawn through the diameter.

588. Def. The *distance* between two points on the surface of a sphere is the arc of a great circle, less than a semi-circumference, drawn between them.



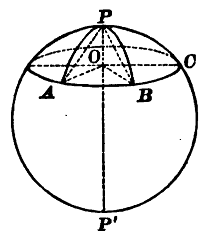
Thus, the distance between the points C and D is the arc CED, and not CFD.

589. Cor. VI. *An arc of a circle may be drawn through any three points on the surface of a sphere.*

For the three points determine a plane, which intersects the surface of the sphere in the arc required.

PROPOSITION VI. THEOREM.

590. *All points in the circumference of a circle of a sphere are equally distant from each of its poles.*



Let P and P' be the poles of the circle ABC of the sphere APC.

To prove that all points in the circumference ABC are equally distant (§ 588) from P, and also from P'.

Let A and B be any two points in the circumference ABC, and draw the arcs of great circles PA and PB.

Draw the axis PP', intersecting the plane ABC at O.

Draw OA, OB, PA, and PB.

Now O is the centre of the circle ABC. (§ 583.)

Whence, $OA = OB$. (§ 143.)

Therefore, chord PA = chord PB. (§ 407, I.)

Whence, arc PA = arc PB. (§ 157.)

But A and B are *any* two points in the circumference ABC.

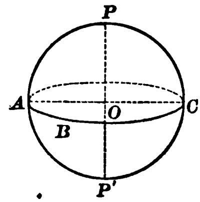
Hence, all points in the circumference ABC are equally distant from P.

In like manner, we may prove that all points in the circumference ABC are equally distant from P'.

591. DEF. The *polar distance* of a circle of a sphere is the distance (§ 588) from the nearer of its poles to the circumference of the circle.

Thus, the polar distance of the circle ABC is the arc PA.

592. Cor. *The polar distance of a great circle is a quadrant.*



Let PA be the polar distance of the great circle ABC.

To prove PA a quadrant (§ 146).

Let O be the centre of the sphere, and draw OA and OP.

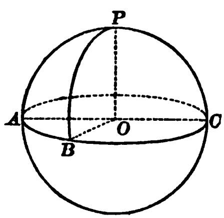
Then, $\angle POA$ is a right angle. (§ 398.)

Therefore, the arc PA is a quadrant. (§ 191.)

593. Sch. The term *quadrant*, in Spherical Geometry, usually signifies a quadrant of a great circle.

Proposition VII. Theorem.

594. *If a point on the surface of a sphere lies at a quadrant's distance from each of two points in the arc of a great circle, it is the pole of that arc.*



Let P be a point on the surface of the sphere AC, at a quadrant's distance from each of the points A and B.

To prove P the pole of the arc of a great circle AB.

Draw the radii OA, OB, and OP.

Then since the arcs PA and PB are quadrants, the angles POA and POB are right angles. (§ 191.)

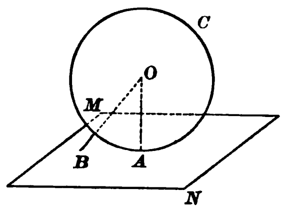
Then, PO is perpendicular to the plane OAB. (§ 400.)

Therefore, P is the pole of the arc AB.

NOTE. **If the two points lie at the extremities of a diameter, the theorem is not necessarily true; for *P* is the pole of only one of the great circles which can be drawn through the points *A* and *C*.**

PROPOSITION VIII. THEOREM.

595. *A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*



Let the plane MN be perpendicular to the radius OA of the sphere AC at its extremity A.

To prove MN tangent to the sphere.

Let B be any point of MN except A, and draw OB.

Then, $OB > OA$. (§ 402.)

Whence, B lies without the sphere.

Then every point of MN except A lies without the sphere, and MN is tangent to the sphere. (§ 579.)

596. COR. (Converse of Prop. VIII.) *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

Let the plane MN be tangent to the sphere AC.

To prove that MN is perpendicular to the radius OA drawn to the point of contact.

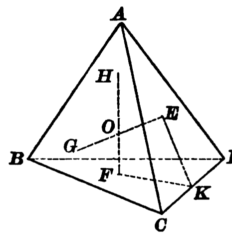
If MN is tangent to the sphere at A, every point of MN except A lies without the sphere.

Then OA is the shortest line that can be drawn from O to MN.

Whence, OA is perpendicular to MN. (§ 402.)

Proposition IX. Theorem.

597. *A sphere may be circumscribed about any tetraedron.*



Let $ABCD$ be any tetraedron.

To prove that a sphere may be circumscribed about it.

Draw EK in the face ACD, and FK in the face BCD, perpendicular to CD at its middle point K.

Let E and F be the centres of the circumscribed circles of the triangles ACD and BCD (§ 222).

Draw EG and FH perpendicular to the planes ACD and BCD.

Now the plane determined by EK and FK is perpendicular to CD. (§ 400.)

Therefore, this plane is perpendicular to each of the planes ACD and BCD. (§ 444.)

Then EG, being perpendicular to the plane ACD, lies in the plane determined by EK and FK. (§ 441.)

In like manner, FH lies in this plane.

Therefore, EG and FH must meet at some point, as O.

Since O is in the perpendicular EG, it is equally distant from A, C, and D. (§ 407, I.)

And since it is in the perpendicular FH, it is equally distant from B, C, and D.

Hence, O is equally distant from A, B, C, and D; and the sphere described with O as a centre, and OA as a radius, will be circumscribed about the tetraedron.

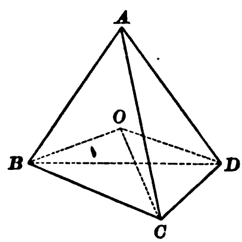
598. Cor. I. *But one sphere can be circumscribed about a given tetraedron.*

For the centre of any circumscribed sphere must lie in the perpendicular drawn to each face at the centre of its circumscribed circle. (§ 408.)

599. Cor. II. *The planes perpendicular to the edges of a tetraedron at their middle points meet in a common point.*

Proposition X. Theorem.

600. *A sphere may be inscribed in any tetraedron.*



Let $ABCD$ be any tetraedron.

To prove that a sphere may be inscribed in it.

Draw the planes OBC, OCD, and ODB, bisecting the diedrals $ABCD$, $ACDB$, and $ADBC$, respectively.

Then since O is in the plane OBC, it is equally distant from the faces ABC and BCD. (§ 446.)

In like manner, O is equally distant from the faces ACD and BCD, and from the faces ABD and BCD.

Hence, O is equally distant from the four faces of the tetraedron; and the sphere described with O as a centre, and the perpendicular from O to either face as a radius, will be inscribed in the tetraedron.

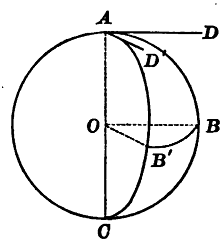
601. Cor. *The planes bisecting the diedrals of a tetraedron meet in a common point.*

602. Definitions. The *angle* between two intersecting curves is the angle included between tangents to the curves at their common point.

A *spherical angle* is the angle between two intersecting arcs of great circles.

Proposition XI. Theorem.

603. *A spherical angle is measured by the arc of a great circle described with its vertex as a pole, included between its sides produced if necessary.*



Let ABC and $AB'C$ be arcs of great circles on the surface of the sphere AC whose centre is O.

Draw AD and AD' tangent to ABC and $AB'C$.

Then DAD' is the angle between the arcs ABC and $AB'C$. (§ 602.)

Draw the diameter AC; also, draw OB and OB' in the planes ABC and $AB'C$ perpendicular to AC, and let their plane intersect the surface of the sphere in the arc BB'.

To prove that $\angle DAD'$ is measured by the arc BB'.

$\angle DAD'$ is the plane angle of the diedral $BACB'$. (§ 170.)

Whence, $\angle DAD' = \angle BOB'$. (§ 429.)

But $\angle BOB'$ is measured by the arc BB'. (§ 192.)

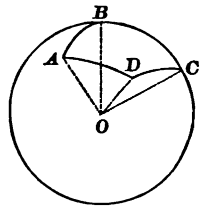
Therefore, $\angle DAD'$ is measured by the arc BB'.

604. Cor. *The angle between two arcs of great circles is the plane angle of the diedral formed by their planes.*

SPHERICAL POLYGONS AND SPHERICAL PYRAMIDS.

Definitions.

605. A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of *great circles*; as *ABCD*.



The bounding arcs are called the *sides* of the spherical polygon.

The *angles* of the spherical polygon are the spherical angles (§ 602) formed by the adjacent sides; and their vertices are called the *vertices* of the spherical polygon.

A *diagonal* is an arc of a great circle joining any two vertices which are not consecutive.

606. The planes of the sides of a spherical polygon form a polyedral, *O–ABCD*, whose vertex is the centre of the sphere, and whose face angles *AOB*, *BOC*, etc., are measured by the sides *AB*, *BC*, etc., of the spherical polygon (§ 192).

A spherical polygon is called *convex* when its corresponding polyedral is convex (§ 456).

607. Since the sum of the face angles of any convex polyedral is less than four right angles (§ 463), the sum of their measures is less than a circumference.

That is, *the sum of the sides of a convex spherical polygon is less than the circumference of a great circle.*

608. A *spherical pyramid* is the solid bounded by a spherical polygon and the planes of its sides; as *O–ABCD* (§ 605).

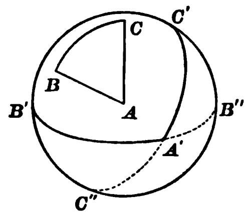
The centre of the sphere is called the *vertex* of the spherical pyramid, and the spherical polygon is called its *base.*

609. The sides of a spherical polygon, being arcs, are usually measured in *degrees.*

610. A *spherical triangle* is a spherical polygon of three sides.

It is called *isosceles, equilateral,* or *right-angled* in the same cases as a plane triangle.

611. If, with the vertices of a spherical triangle as poles, arcs of great circles be described, a spherical triangle is formed which is called the *polar triangle* of the first.



Thus, if A, B, and C are the poles of the arcs $B'C'$, $C'A'$, and $A'B'$, then $A'B'C'$ is the polar triangle of ABC.

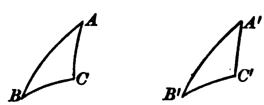
612. The circumferences of which $B'C'$, $C'A'$, and $A'B'$ are arcs, form by their intersection *eight* spherical triangles.

Four of these, i.e., $A'B'C'$, $A'B'C''$, $A'B''C'$, and $A'B''C''$, lie on the hemisphere represented in the figure, and the others lie on the opposite hemisphere.

Of these eight spherical triangles, that is the polar triangle in which the vertex A' homologous to A, lies on the same side of BC as the vertex A; and similarly for the remaining vertices.

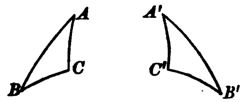
613. Two spherical polygons are *equal* when they can be applied one to the other so as to coincide throughout.

614. Two spherical polygons are equal when the sides and angles of one are equal respectively to the homologous sides and angles of the other, if the equal parts are arranged *in the same order.*



Thus, the spherical triangles ABC and $A'B'C'$ are equal if the sides AB, BC, and CA are equal to $A'B'$, $B'C'$, and $C'A'$, respectively, and the angles A, B, and C to the angles A', B', and C'; for they can evidently be applied one to the other so as to coincide throughout.

615. Two spherical polygons are said to be ***symmetrical*** when the sides and angles of one are equal respectively to the homologous sides and angles of the other, if the equal parts are arranged *in the reverse order.*

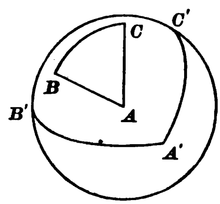


Thus, the spherical triangles ABC and $A'B'C'$ are symmetrical if the sides AB, BC, and CA are equal to $A'B'$, $B'C'$, and $C'A'$, respectively, and the angles A, B, and C to the angles A', B', and C'.

616. It is evident that in general two symmetrical spherical triangles cannot be applied one to the other so as to coincide throughout. (Compare § 636.)

Proposition XII. Theorem.

617. *If one spherical triangle is the polar triangle of another, then the second spherical triangle is the polar triangle of the first.*



Let $A'B'C'$ be the polar triangle of ABC.

To prove that ABC is the polar triangle of $A'B'C'$.

Now B is the pole of the arc $A'C'$. (§ 611.)
Whence, A' lies at a quadrant's distance from B. (§ 592.)
Again, C is the pole of the arc $A'B'$.
Whence, A' lies at a quadrant's distance from C.
Therefore, A' is the pole of the arc BC. (§ 594.)

In like manner, it may be proved that B' is the pole of the arc CA, and C' of the arc AB.

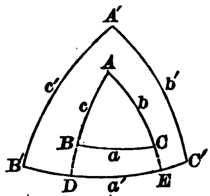
Then, ABC is the polar triangle of $A'B'C'$.

For the homologous vertices A and A' lie on the same side of $B'C'$ (§ 612), and similarly for the remaining vertices.

618. Def. Two spherical triangles, each of which is the polar triangle of the other, are called *polar triangles.*

Proposition XIII. Theorem.

619. *In two polar triangles, each angle of one is measured by the supplement of the side lying opposite the homologous angle of the other.*



Let a, b, c, and a', b', c' denote the sides, expressed in degrees, and A, B, C, and A', B', C' the angles, also expressed in degrees, of the polar triangles ABC and $A'B'C'$.

To prove

$$A = 180° - a', \quad B = 180° - b', \quad C = 180° - c',$$
$$A' = 180° - a, \quad B' = 180° - b, \quad C' = 180° - c.$$

Produce the arcs AB and AC to meet the arc $B'C'$ at D and E.

Then since B' is the pole of the arc AE, and C' of the arc AD, the arcs $B'E$ and $C'D$ are quadrants. (§ 592.)

Therefore, arc $B'E$ + arc $C'D = 180°$.

That is, arc DE + arc $B'C' = 180°$.

But A is the pole of the arc $B'C'$.

Whence, the angle A is measured by the arc DE. (§ 603.)

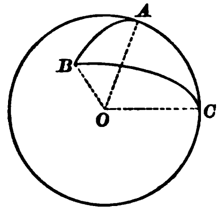
Therefore, $A + a' = 180°$.

Whence, $A = 180° - a'$.

In like manner, the theorem may be proved for any angle of either triangle.

Proposition XIV. Theorem.

620. *The sum of any two sides of a spherical triangle is greater than the third side.*



Let ABC be a spherical triangle on the surface of a sphere whose centre is O.

To prove $AB + AC > BC$.

Draw OA, OB, and OC; then in the triedral $O\text{–}ABC$,

$$\angle AOB + \angle AOC > \angle BOC. \qquad (\S\ 462.)$$

But the sides AB, AC, and BC are the measures of the angles AOB, AOC, and BOC, respectively. (§ 192.)

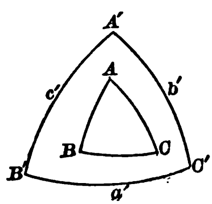
Therefore, $AB + AC > BC$.

In like manner, we may prove

$$AB + BC > AC, \text{ and } AC + BC > AB.$$

Proposition XV. Theorem.

621. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*



Let ABC be a spherical triangle.

To prove $A + B + C > 180°$, and $< 540°$.

Let $A'B'C'$ be the polar triangle of ABC, and denote its sides by a', b', and c'.

Then, $A = 180° - a'$,

$B = 180° - b'$,

and $C = 180° - c'$. (§ 619.)

Adding these equations, we have

$$A + B + C = 540° - (a' + b' + c'). \qquad (1)$$

Whence, $A + B + C < 540°$.

Again, the sum of the sides of the spherical triangle $A'B'C'$ is less than the circumference of a great circle. (§ 607.)

That is, $a' + b' + c' < 360°$.

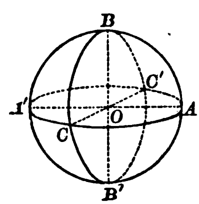
Whence by (1), $A + B + C > 180°$.

622. Cor. I. *A spherical triangle may have one, two, or three right angles, or one, two, or three obtuse angles.*

623. Def. A spherical triangle having two right angles is called a *bi-rectangular triangle.*

A spherical triangle having three right angles is called a *tri-rectangular triangle.*

624. Cor. II. *If three planes be passed through the centre of a sphere in such a way that each is perpendicular to the other two, the surface is divided into eight equal tri-rectangular triangles.*



For each angle of either spherical triangle is a right angle.

Also, each side of either triangle is a quadrant. (§ 191.)

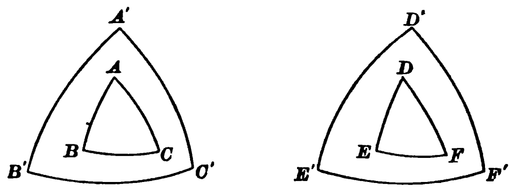
Hence, the spherical triangles are all equal. (§ 614.)

625. Cor. III. *The surface of a sphere is eight times the surface of a tri-rectangular triangle.*

626. Def. Two spherical polygons on the same sphere, or equal spheres, are said to be *mutually equilateral*, or *mutually equiangular*, when the sides or angles of one are equal respectively to the homologous sides or angles of the other, whether taken in the same or in the reverse order.

Proposition XVI. Theorem.

627. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular, their polar triangles are mutually equilateral.*



Let ABC and DEF be mutually equiangular spherical triangles on the same sphere, or on equal spheres; the angles A and D being homologous.

To prove that their polar triangles, $A'B'C'$ and $D'E'F'$, are mutually equilateral.

The angles A and D are measured by the supplements of the sides $B'C'$ and $E'F'$, respectively. (§ 619.)

But by hypothesis, $\angle A = \angle D$.

Whence, $B'C' = E'F'$. (§ 33, 2.)

In like manner, any two homologous sides of $A'B'C'$ and $D'E'F'$ may be proved equal.

Therefore, $A'B'C'$ and $D'E'F'$ are mutually equilateral.

628. Cor. (Converse of Prop. XVI.) *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral, their polar triangles are mutually equiangular.*

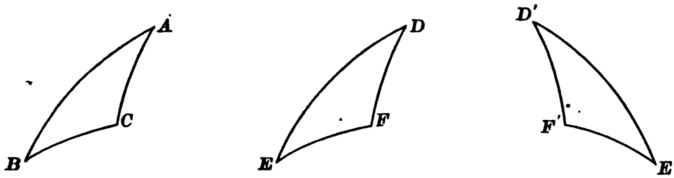
(The proof is left to the student; compare § 627.)

PROPOSITION XVII. THEOREM.

629. *If two spherical triangles on the same sphere, or equal spheres, have two sides and the included angle of one equal respectively to two sides and the included angle of the other,*

I. *They are equal if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in the reverse order.*



I. Let ABC and DEF be spherical triangles on the same sphere, or equal spheres, having

$$AB = DE,\ AC = DF, \text{ and } \angle A = \angle D.$$

To prove ABC and DEF equal.

Superpose ABC upon DEF so that $\angle A$ shall coincide with $\angle D$; the side AB falling upon DE, and AC upon DF.

Then, since $AB = DE$ and $AC = DF$, B will fall at E, and C at F; and the side BC will coincide with EF. (§ 587.)

Hence, ABC and DEF coincide throughout, and are equal.

II. Let ABC and $D'E'F'$ be spherical triangles on the same sphere, or equal spheres, having

$$AB = D'E',\ AC = D'F', \text{ and } \angle A = \angle D'.$$

To prove ABC and $D'E'F'$ symmetrical.

Construct the spherical triangle DEF symmetrical to $D'E'F'$, having $DE = D'E'$, $DF = D'F'$, and $\angle D = \angle D'$.

Then in the spherical triangles ABC and DEF, we have

$$AB = DE,\ AC = DF, \text{ and } \angle A = \angle D.$$

Whence, ABC and DEF are equal. (§ 629, I.)

Therefore, ABC is symmetrical to $D'E'F'$.

PROPOSITION XVIII. THEOREM.

630. *If two spherical triangles on the same sphere, or equal spheres, have a side and two adjacent angles of one equal respectively to a side and two adjacent angles of the other,*

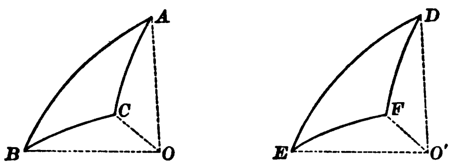
I. *They are equal if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in the reverse order.*

(The proof is left to the student; compare § 629.)

PROPOSITION XIX. THEOREM.

631. *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral, they are mutually equiangular.*



Let ABC and DEF be mutually equilateral spherical triangles on equal spheres, the sides BC and EF being homologous.

To prove ABC and DEF mutually equiangular.

Let O and O' be the centres of the respective spheres.

Draw OA, OB, OC, $O'D$, $O'E$, and $O'F$.

The triedrals $O\text{-}ABC$ and $O'\text{-}DEF$ have their homologous face angles equal. (§ 192.)

Therefore, diedral OA = diedral $O'D$. (§ 465.)

But the spherical angles BAC and EDF are the plane angles of the diedrals OA and $O'D$. (§ 604.)

Whence, $\angle BAC = \angle EDF$. (§ 432.)

In like manner, any two homologous angles of ABC and DEF may be proved equal.

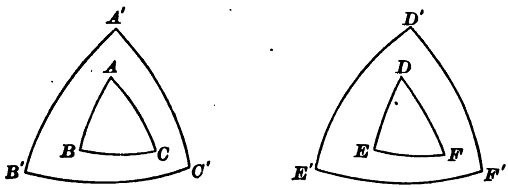
Whence, ABC and DEF are mutually equiangular.

632. Cor. *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral,*

1. *They are equal if the equal parts occur in the same order.*
2. *They are symmetrical if the equal parts occur in the reverse order.*

Proposition XX. Theorem.

633. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular, they are mutually equilateral.*



Let ABC and DEF be mutually equiangular spherical triangles on the same sphere, or equal spheres.

To prove ABC and DEF mutually equilateral.

Let $A'B'C'$ be the polar triangle of ABC, and $D'E'F'$ of DEF.

Then since ABC and DEF are mutually equiangular, $A'B'C'$ and $D'E'F'$ are mutually equilateral. (§ 627.)

Then $A'B'C'$ and $D'E'F'$ are mutually equiangular. (§ 631.)

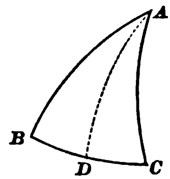
Therefore, their polar triangles, ABC and DEF, are mutually equilateral. (§ 627.)

634. Cor. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular,*

1. *They are equal if the equal parts occur in the same order.*
2. *They are symmetrical if the equal parts occur in the reverse order.*

Proposition XXI. Theorem.

635. *In an isosceles spherical triangle, the angles opposite the equal sides are equal.*



In the spherical triangle ABC, let $AB = AC$.

To prove $\angle B = \angle C$.

Let AD be an arc of a great circle bisecting the side BC.

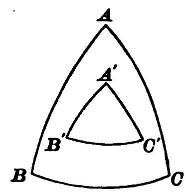
Then in the spherical triangles ABD and ACD, the side AD is common; also, $AB = AC$, and $BD = CD$.

Then ABD and ACD are mutually equiangular. (§ 631.)

Whence, $\angle B = \angle C$.

636. Cor. I. *Two symmetrical* (§ 615) *isosceles spherical triangles are equal*; for they can be applied one to the other so as to coincide throughout.

637. Cor. II. (Converse of Prop. XXI.) *If two angles of a spherical triangle are equal, the sides opposite are equal.*



In the spherical triangle ABC, let

$$\angle B = \angle C.$$

To prove $AB = AC$.

Let $A'B'C'$ be the polar triangle of ABC.

Then, $A'B'$ is the supplement of $\angle C$, and $A'C'$ of $\angle B$. (§ 619.)

Whence, $A'B' = A'C'$. (§ 33, 2.)

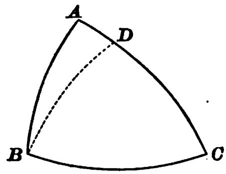
Therefore, $\angle C' = \angle B'$. (§ 635.)

But AB is the supplement of $\angle C'$, and AC of $\angle B'$.

Whence, $AB = AC$.

PROPOSITION XXII. THEOREM.

638. *In any spherical triangle, the greater side lies opposite the greater angle.*



In the spherical triangle ABC, let $\angle ABC$ be $> \angle C$.

To prove $AC > AB$.

Let BD be an arc of a great circle making $\angle CBD = \angle C$.

Then, $BD = CD$. (§ 637.)

But, $AD + BD > AB$. (§ 620.)

Therefore, $AD + CD > AB$.

That is, $AC > AB$.

639. COR. (Converse of Prop. XXII.) *In any spherical triangle, the greater angle lies opposite the greater side.*

In the spherical triangle ABC, let AC be $> AB$.

To prove $\angle ABC > \angle C$.

If $\angle ABC$ were $< \angle C$, AC would be $< AB$. (§ 638.)

And if $\angle ABC$ were equal to $\angle C$, AC would be equal to AB. (§ 637.)

But each of these conclusions is contrary to the hypothesis that AC is $> AB$.

Hence, $\angle ABC > \angle C$.

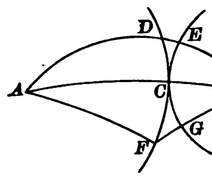
EXERCISES.

1. If the sides of a spherical triangle are 77°, 123°, and 95°, how many degrees are there in each angle of its polar triangle?

2. If the angles of a spherical triangle are 86°, 131°, and 68°, how many degrees are there in each side of its polar triangle?

PROPOSITION XXIII. THEOREM.

640. *The shortest line on the surface of a sphere between two given points is the arc of a great circle, not greater than a semi-circumference, which joins the points.*



Let AB be an arc of a great circle, not greater than a semi-circumference, which joins the given points A and B.

To prove AB the shortest line on the surface of the sphere between A and B.

Let C be any point in the arc AB.

Let DCF and ECG be arcs of small circles, having A and B respectively as poles, and AC and BC as polar distances.

The arcs DCF and ECG have only the point C common.

For let F be any other point in the arc DCF, and draw the arcs of great circles AF and BF.

Then, $AF + BF > AC + BC$. (§ 620.)

Subtracting arc AF from the first member of the inequality, and its equal AC from the second member, we have

$$BF > BC.$$

Therefore, F lies without the small circle ECG, and the arcs DCF and ECG have only the point C common.

We will next prove that the shortest line on the surface of the sphere from A to B must pass through C.

Let $ADEB$ be any line drawn on the surface of the sphere between A and B, not passing through C, and cutting the arcs DCF and ECG at D and E, respectively.

Then, whatever the nature of the line AD, it is evident that an equal line can be drawn from A to C.

In like manner, whatever the nature of the line *BE*, an equal line can be drawn from *B* to *C*.

Hence, a line can be drawn from *A* to *B* passing through *C*, equal to the sum of the lines *AD* and *BE*, and consequently less than the line *ADEB* by the portion *DE*.

Therefore, no line which does not pass through *C* can be the shortest line between *A* and *B*.

But *C* is *any* point in the arc *AB*.

Hence the shortest line from *A* to *B* must pass through *every* point of *AB*.

That is, the arc of a great circle *AB* is the shortest line which can be drawn on the surface of the sphere between *A* and *B*.

EXERCISES.

3. Any point in the arc of a great circle bisecting a spherical angle is equally distant (§ 588) from the sides of the angle.

[Prove, by §§ 629, II., and 33, 1, the equality of the arcs of great circles joining the point to the poles of the sides of the angle.]

4. State and prove the converse of Ex. 3.

5. The sum of the angles of a spherical hexagon is greater than 8, and less than 12, right angles.

6. The sum of the angles of a spherical polygon of n sides is greater than $2n - 4$, and less than $2n$, right angles.

7. The sides opposite the equal angles of a bi-rectangular triangle are quadrants.

8. State and prove the converse of Ex. 7.

9. Any side of a spherical polygon is less than the sum of the remaining sides.

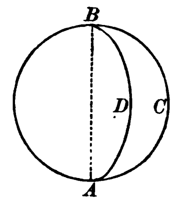
10. The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle point of the base, is perpendicular to the base, and bisects the vertical angle.

11. How many degrees are there in the polar distance of a circle, whose plane is $5\sqrt{2}$ units from the centre of the sphere, the diameter of the sphere being 20 units?

12. The polar distance of a circle of a sphere is 60°. If the diameter of the circle is 6, find the diameter of the sphere, and the distance of the circle from its centre.

MEASUREMENT OF SPHERICAL POLYGONS.

641. Def. A *lune* is a portion of the surface of a sphere bounded by two semi-circumferences of great circles; as *ACBD*.



The *angle* of the lune is the angle included between its bounding arcs.

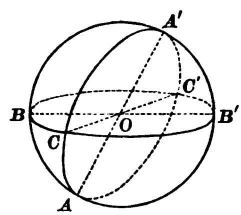
642. It is evident that *two lunes on the same sphere, or equal spheres, are equal when their angles are equal.*

643. A *spherical wedge* is the solid bounded by a lune and the planes of its bounding arcs.

The lune is called the *base* of the spherical wedge.

Proposition XXIV. Theorem.

644. *The spherical triangles corresponding to a pair of vertical triedrals are symmetrical.*



Let AOA', BOB', and COC' be diameters of the sphere AC.

To prove the spherical triangles ABC and $A'B'C'$ symmetrical.

The angles AOB, BOC, and COA are equal respectively to the angles $A'OB'$, $B'OC'$, and $C'OA'$. (§ 39.)

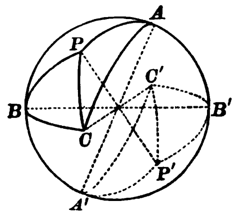
Then, $AB = A'B'$, $BC = B'C'$, and $CA = C'A'$. (§ 192.)

But the equal parts of ABC and $A'B'C'$ are arranged in the reverse order.

Therefore, ABC and $A'B'C'$ are symmetrical. (§ 632, 2.)

Proposition XXV. Theorem.

645. *Two symmetrical spherical triangles are equivalent.*



Let AA', BB', and CC' be diameters of the sphere AB.

Then the spherical triangles ABC and $A'B'C'$ are symmetrical. (§ 644.)

To prove area $ABC =$ area $A'B'C'$.

Let P be the pole of the small circle passing through the points A, B, and C, and draw the arcs of great circles PA, PB, and PC.

Then, $PA = PB = PC$. (§ 590.)

Draw the diameter of the sphere PP'.

Also, draw the arcs of great circles $P'A'$, $P'B'$, and $P'C'$.

Then the spherical triangles PAB and $P'A'B'$ are symmetrical. (§ 644.)

But the spherical triangle PAB is isosceles.

Therefore, PAB is equal to $P'A'B'$. (§ 636.)

In like manner, we may prove PBC equal to $P'B'C'$, and PCA equal to $P'C'A'$.

Then the sum of the areas of the triangles PAB, PBC, and PCA is equal to the sum of the areas of $P'A'B'$, $P'B'C'$, and $P'C'A'$.

That is, area $ABC =$ area $A'B'C'$.

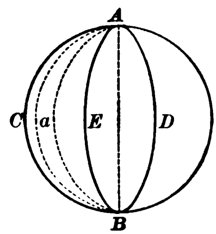
646. Sch. If P and P' fall without the spherical triangles ABC and $A'B'C'$, we should take the sum of the areas of two isosceles spherical triangles, diminished by the area of a third.

Proposition XXVI. Theorem.

647. *Two lunes on the same sphere, or equal spheres, are to each other as their angles.*

Note. The word "*lune*," in the above statement, signifies the *area* of the lune.

Case I. *When the angles are commensurable.*



Let $ACBD$ and $ACBE$ be two lunes, whose angles CAD and CAE are commensurable.

To prove $$\frac{ACBD}{ACBE} = \frac{\angle\ CAD}{\angle\ CAE}.$$

Let CAa be a common measure of the angles CAD and CAE, and let it be contained 5 times in CAD, and 3 times in CAE.

Then, $$\frac{\angle\ CAD}{\angle\ CAE} = \frac{5}{3}. \qquad (1)$$

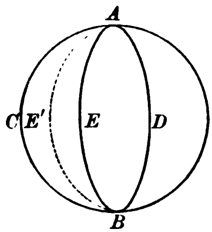
Producing the several arcs of division of the angle CAD to B, the lune $ACBD$ will be divided into 5 parts, and the lune $ACBE$ into 3 parts, all of which parts will be equal. (§ 642.)

Then, $$\frac{ACBD}{ACBE} = \frac{5}{3}. \qquad (2)$$

From (1) and (2), we have,

$$\frac{ACBD}{ACBE} = \frac{\angle\ CAD}{\angle\ CAE}.$$

CASE II. *When the angles are incommensurable.*



Let $ACBD$ and $ACBE$ be two lunes, whose angles CAD and CAE are incommensurable.

To prove $$\frac{ACBD}{ACBE} = \frac{\angle\, CAD}{\angle\, CAE}.$$

Let $\angle\, CAD$ be divided into any number of equal parts, and let one of these parts be applied to $\angle\, CAE$ as a measure.

Since CAD and CAE are incommensurable, a certain number of the parts will extend from AC to AE', leaving a remainder $E'AE$ less than one of the parts.

Produce the arc AE' to B.

Then, $$\frac{ACBD}{ACBE'} = \frac{\angle\, CAD}{\angle\, CAE'}. \quad \text{(§ 647, Case I.)}$$

Now let the number of subdivisions of the angle CAD be indefinitely increased.

Then the magnitude of each part will be indefinitely diminished, and the remainder $E'AE$ will approach the limit 0.

Then, $\dfrac{ACBD}{ACBE'}$ will approach the limit $\dfrac{ACBD}{ACBE}$,

and $\dfrac{\angle\, CAD}{\angle\, CAE'}$ will approach the limit $\dfrac{\angle\, CAD}{\angle\, CAE}$.

By the Theorem of Limits, these limits are equal. (§ 188.)

Whence, $$\frac{ACBD}{ACBE} = \frac{\angle\, CAD}{\angle\, CAE}.$$

648. Cor. I. *The surface of a lune is to the surface of the sphere as the angle of the lune is to four right angles.*

For the surface of a sphere may be regarded as a lune whose angle is equal to four right angles.

649. Cor. II. Let L denote the area of a lune; A the numerical measure of its angle referred to a right angle as the unit; and T the area of a tri-rectangular triangle.

Then the area of the surface of the sphere is $8\,T$. (§ 625.)

Whence, $$\frac{L}{8\,T} = \frac{A}{4}. \qquad (§ 648.)$$

That is, $$L = 2\,A \times T.$$

Hence, *if the unit of measure for angles is the right angle, the area of a lune is equal to twice its angle, multiplied by the area of a tri-rectangular triangle.*

Thus, if the area of the surface of a sphere is 72, the area of a tri-rectangular triangle is $\frac{1}{8}$ of 72, or 9.

Then, if the angle of a lune on this sphere is 50°, or $\frac{5}{9}$ of a right angle, its area is $\frac{10}{9}$ of 9, or 10.

650. Sch. It may be proved, exactly as in § 647, that

The volume of a spherical wedge is to the volume of the sphere as the angle of the lune which forms its base is to four right angles.

It follows from the above that

If the unit of measure for angles is the right angle, the volume of a spherical wedge is equal to twice the angle of the lune which forms its base, multiplied by the volume of a tri-rectangular pyramid.

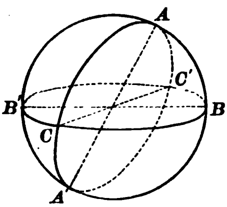
Note. A *tri-rectangular pyramid* is a spherical pyramid whose base is a tri-rectangular triangle.

651. Def. The *spherical excess* of a spherical triangle is the excess of the sum of its angles above two right angles.

Thus, if the angles of a spherical triangle are 65°, 80°, and 95°, its spherical excess is 65° + 80° + 95° − 180°, or 60°.

PROPOSITION XXVII. THEOREM.

652. *If the unit of measure for angles is the right angle, the area of a spherical triangle is equal to its spherical excess, multiplied by the area of a tri-rectangular triangle.*



Let A, B, and C denote the numerical measures of the angles of the spherical triangle ABC referred to a right angle as the unit; and T the area of a tri-rectangular triangle.

To prove area $ABC = (A + B + C - 2) \times T$.

Complete the circumferences $ABA'B'$, $ACA'C'$, and $BCB'C'$; and draw the diameters AA', BB', and CC'.

Then since $ABA'C$ is a lune whose angle is A, we have

area ABC + area $A'BC = 2\,A \times T$ (§ 649). (1)

And since $BAB'C$ is a lune whose angle is B,

area ABC + area $AB'C = 2\,B \times T$. (2)

Now $A'B'C$ and ABC' are symmetrical. (§ 644.)

Whence, area $A'B'C$ = area ABC'. (§ 645.)

Adding area ABC to both members, we have

area ABC + area $A'B'C$ = area of lune $CBC'A$

$= 2\,C \times T$. (3)

Adding (1), (2), and (3), and observing that the sum of the areas of ABC, $A'BC$, $AB'C$, and $A'B'C$ is equal to the area of the surface of a hemisphere, or $4\,T$, we have

$$2 \text{ area } ABC + 4\,T = (2\,A + 2\,B + 2\,C) \times T.$$

Then, area $ABC + 2\,T = (A + B + C) \times T$.

Or, area $ABC = (A + B + C - 2) \times T$.

653. Sch. I. Let it be required to find the area of a spherical triangle whose angles are 105°, 80°, and 95°, on a sphere the area of whose surface is 144 sq. in.

The spherical excess of the spherical triangle is 100°, or $\frac{10}{9}$ referred to a right angle as the unit.

And the area of a tri-rectangular triangle is $\frac{1}{8}$ of 144, or 18 sq. in.

Hence, the required area is $\frac{10}{9}$ of 18, or 20 sq. in.

654. Sch. II. It may be proved, as in § 652, that

If the unit of measure for angles is the right angle, the volume of a spherical pyramid is equal to the spherical excess of its base, multiplied by the volume of a tri-rectangular pyramid.

EXERCISES.

13. Find the area of a spherical triangle whose angles are 103°, 112°, and 127°, on a sphere the area of whose surface is 160.

14. Find the volume of a triangular spherical pyramid the angles of whose base are 92°, 119°, and 134°; the volume of the sphere being 192.

15. What is the volume of a spherical wedge the angle of whose base is 127° 30′, if the volume of the sphere is 112?

16. The area of a lune is $28\frac{4}{5}$ sq. in. If the area of the surface of the sphere is 120 sq. in., what is the angle of the lune?

17. What is the ratio of the areas of two spherical triangles on the same sphere, whose angles are 94°, 135°, and 146°, and 87°, 105°, and 118°, respectively?

18. The area of a spherical triangle two of whose angles are 78° and 99°, is $34\frac{1}{8}$. If the area of the surface of the sphere is 234, what is the other angle?

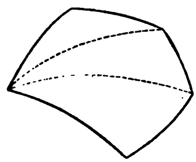
19. The volume of a triangular spherical pyramid the angles of whose base are 105°, 126°, and 147°, is $60\frac{1}{2}$; what is the volume of the sphere?

20. If two straight lines are tangent to a sphere at the same point, their plane is tangent to the sphere.

21. The sum of the arcs of great circles drawn from any point within a spherical triangle to the extremities of any side, is less than the sum of the other two sides of the triangle.

Proposition XXVIII. Theorem.

655. *If the unit of measure for angles is the right angle, the area of any spherical polygon is equal to the sum of its angles, diminished by as many times two right angles as the figure has sides less two, multiplied by the area of a tri-rectangular triangle.*



Let K denote the area of any spherical polygon; n the number of its sides; s the sum of its angles referred to a right angle as the unit; and T the area of a tri-rectangular triangle.

To prove $\quad K = [s - 2(n - 2)] \times T.$

The spherical polygon may be divided into spherical triangles by drawing diagonals from any vertex; the number of such spherical triangles being equal to the number of sides of the spherical polygon, less two.

Now the area of each spherical triangle is equal to the sum of its angles, less two right angles, multiplied by T. (§ 652.)

Hence, the sum of the areas of the spherical triangles is equal to the sum of their angles, diminished by as many times two right angles as there are triangles, multiplied by T.

But the number of triangles is $n - 2$.

Therefore, the area of the spherical polygon is equal to the sum of its angles, diminished by $n - 2$ times two right angles, multiplied by T.

That is, $\quad K = [s - 2(n - 2)] \times T.$

656. SCH. It may be proved, as in § 655, that

If the unit of measure for angles is the right angle, the volume of any spherical pyramid is equal to the sum of the angles of its base, diminished by as many times two right angles as the base has sides less two, multiplied by the volume of a tri-rectangular pyramid.

657. COR. Let P denote the volume of a spherical pyramid; and let K denote the area of the base, n the number of its sides, and s the sum of its angles referred to a right angle as the unit.

Let V represent the volume of the sphere; S the area of its surface; T the area of a tri-rectangular triangle; and T' the volume of a tri-rectangular pyramid.

Then, $\quad P = [s - 2(n-2)] \times T', \quad$ (§ 656.)

and $\quad K = [s - 2(n-2)] \times T. \quad$ (§ 655.)

Also, $\quad V = 8\,T'$, and $S = 8\,T$.

Whence by division,

$$\frac{P}{K} = \frac{T'}{T}, \text{ and } \frac{V}{S} = \frac{T'}{T}.$$

Therefore, $$\frac{P}{K} = \frac{V}{S}.$$

That is, *the volume of a spherical pyramid is to its base as the volume of the sphere is to its surface.*

EXERCISES.

22. Find the area of a spherical hexagon whose angles are 120°, 139°, 148°, 155°, 162°, and 167°, on a sphere the area of whose surface is 280.

23. Find the volume of a pentagonal spherical pyramid the angles of whose base are 109°, 128°, 137°, 153°, and 158°; the volume of the sphere being 180.

24. The arcs of great circles bisecting the angles of a spherical triangle meet in a point equally distant from the sides of the triangle. (Exs. 3, 4, p. 337.)

25. A circle may be inscribed in any spherical triangle.

26. State and prove the theorem for spherical triangles analogous to Prop. IX., Book I.

27. State and prove the theorem for spherical triangles analogous to Prop. V., Book I.

28. State and prove the theorem for spherical triangles analogous to Prop. LI., Book I.

29. The volume of a quadrangular spherical pyramid, the angles of whose base are 110°, 122°, 135°, and 146°, is $12\frac{3}{4}$ cu. ft. What is the volume of the sphere?

30. The area of a spherical pentagon, four of whose angles are 112°, 131°, 138°, and 168°, is 27. If the area of the surface of the sphere is 120, what is the other angle?

31. If the side AB of a spherical triangle ABC is equal to a quadrant, and the side BC is less than a quadrant, prove that $\angle A$ is less than 90°.

32. If PA, PB, and PC are three equal arcs of great circles drawn from a point P to the circumference of a great circle ABC, prove that P is the pole of ABC.

33. The spherical polygons corresponding to a pair of vertical polyedrals are symmetrical.

34. Either angle of a spherical triangle is greater than the difference between 180° and the sum of the other two angles.

35. If a polyedron be circumscribed about each of two equal spheres, the volumes of the polyedrons are to each other as the areas of their surfaces.

36. If ABC and $A'B'C'$ are a pair of polar triangles on a sphere whose centre is O, prove that the radius OA' is perpendicular to the plane OBC.

37. The intersection of two spheres is a circle, whose centre lies in the line joining the centres of the spheres, and whose plane is perpendicular to this line.

38. The distance between the centres of two spheres, whose radii are 25 in. and 17 in., respectively, is 28 in. Find the diameter of their circle of intersection, and the distance of its plane from the centre of each sphere.

Book IX.

MEASUREMENT OF THE CYLINDER, CONE, AND SPHERE.

THE CYLINDER.

Definitions.

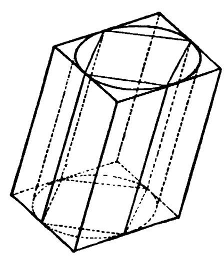
658. A prism is said to be *inscribed in a cylinder* when its bases are inscribed in the bases of the cylinder.

A prism is said to be *circumscribed about a cylinder* when its bases are circumscribed about the bases of the cylinder.

The *lateral area* of a cylinder is the area of its lateral surface.

A *right section* of a cylinder is a section perpendicular to its elements.

659. If a regular polygon be inscribed in, or circumscribed about, a circle, and the number of its sides be indefinitely increased, its perimeter and area approach the circumference and area of the circle respectively as limits (§ 363).



Hence, *if a prism whose base is a regular polygon be inscribed in, or circumscribed about a circular cylinder* (§ 553), *and the number of its faces be indefinitely increased,*

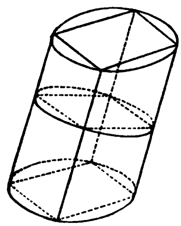
1. *The lateral area of the prism approaches the lateral area of the cylinder as a limit.*

2. *The volume of the prism approaches the volume of the cylinder as a limit.*

3. *The perimeter of a right section of the prism approaches the perimeter of a right section of the cylinder as a limit.*

Proposition I. Theorem.

660. *The lateral area of a circular cylinder is equal to the perimeter of a right section, multiplied by an element.*



Let S denote the lateral area, P the perimeter of a right section, and E an element, of a circular cylinder.

To prove $S = P \times E$.

Inscribe in the cylinder a prism whose base is a regular polygon.

Let S' denote its lateral area, and P' the perimeter of a right section.

Then since the lateral edge of the prism is E, we have

$$S' = P' \times E. \qquad (\S\ 485.)$$

Now let the number of faces of the prism be indefinitely increased.

Then, S' approaches the limit S,

and $P' \times E$ approaches the limit $P \times E$.

(§ 659, 1, 3.)

By the Theorem of Limits, these limits are equal. (§ 188.)

Therefore, $S = P \times E$.

661. Cor. I. *The lateral area of a cylinder of revolution is equal to the perimeter of its base multiplied by its altitude.*

662. Cor. II. Let S denote the lateral area, T the total area, H the altitude, and R the radius of the base, of a cylinder of revolution.

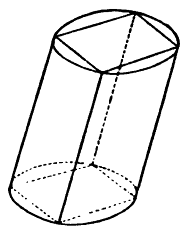
Then, $S = 2\pi RH.$ (§ 368.)

And $T = 2\pi RH + 2\pi R^2$ (§ 371.)

$= 2\pi R(H + R).$

PROPOSITION II. THEOREM.

663. *The volume of a circular cylinder is equal to the product of its base and altitude.*



Let V denote the volume, B the area of the base, and H the altitude, of a circular cylinder.

To prove $V = B \times H.$

Inscribe in the cylinder a prism whose base is a regular polygon; let V' denote its volume, and B' the area of its base.

Then since the altitude of the prism is H, we have

$V' = B' \times H.$ (§ 507.)

Now let the number of faces of the prism be indefinitely increased.

Then, V' approaches the limit V. (§ 659, 2.)

And $B' \times H$ approaches the limit $B \times H$. (§ 363, II.)

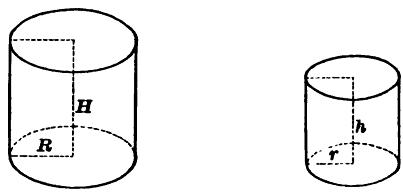
Therefore, $V = B \times H.$ (§ 188.)

664. Cor. Let V denote the volume, H the altitude, and R the radius of the base, of a circular cylinder.

Then, $V = \pi R^2 H.$ (§ 371.)

PROPOSITION III. THEOREM.

665. *The lateral or total areas of two similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*



Let S and s denote the lateral areas, T and t the total areas, V and v the volumes, H and h the altitudes, and R and r the radii of the bases, of two similar cylinders of revolution (§ 555).

To prove $$\frac{S}{s} = \frac{T}{t} = \frac{H^2}{h^2} = \frac{R^2}{r^2},$$

and $$\frac{V}{v} = \frac{H^3}{h^3} = \frac{R^3}{r^3}.$$

The generating rectangles are similar.

Whence, $$\frac{H}{h} = \frac{R}{r} \qquad \text{(§ 253, II.)}$$

$$= \frac{H+R}{h+r}. \qquad \text{(§ 239.)}$$

Then,

$$\frac{S}{s} = \frac{2\pi RH}{2\pi rh} \text{ (§ 662)} = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{H^2}{h^2};$$

$$\frac{T}{t} = \frac{2\pi R(H+R)}{2\pi r(h+r)} \text{ (§ 662)} = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{H^2}{h^2};$$

and

$$\frac{V}{v} = \frac{\pi R^2 H}{\pi r^2 h} \text{ (§ 664)} = \frac{R^2}{r^2} \times \frac{R}{r} = \frac{R^3}{r^3} = \frac{H^3}{h^3}.$$

THE CONE.

Definitions.

666. A pyramid is said to be *inscribed in a cone* when its base is inscribed in the base of the cone, and its vertex coincides with the vertex of the cone.

A pyramid is said to be *circumscribed about a cone* when its base is circumscribed about the base of the cone, and its vertex coincides with the vertex of the cone.

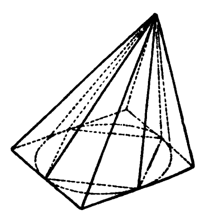
A frustum of a pyramid is said to be *inscribed in a frustum of a cone* when its bases are inscribed in the bases of the frustum of the cone.

A frustum of a pyramid is said to be *circumscribed about a frustum of a cone* when its bases are circumscribed about the bases of the frustum of the cone.

The *lateral area* of a cone, or frustum of a cone, is the area of its lateral surface.

The *slant height* of a cone of revolution is the straight line drawn from the vertex to any point in the circumference of the base.

The *slant height* of a frustum of a cone of revolution is that portion of the slant height of the cone included between the bases of the frustum.



667. It may be proved, exactly as in § 659, that

If a pyramid whose base is a regular polygon be inscribed in, or circumscribed about, a circular cone (§ 565), *and the number of its faces be indefinitely increased,*

1. *The lateral area of the pyramid approaches the lateral area of the cone as a limit.*

2. *The volume of the pyramid approaches the volume of the cone as a limit.*

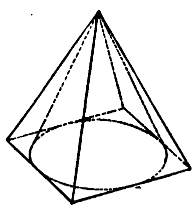
668. It follows from § 667 that

If a frustum of a pyramid whose base is a regular polygon be inscribed in, or circumscribed about, a frustum of a circular cone, and the number of its faces be indefinitely increased,

1. *The lateral area of the frustum of the pyramid approaches the lateral area of the frustum of the cone as a limit.*
2. *The volume of the frustum of the pyramid approaches the volume of the frustum of the cone as a limit.*

Proposition IV. Theorem.

669. *The lateral area of a cone of revolution is equal to the circumference of its base, multiplied by one-half its slant height.*



Let S denote the lateral area, C the circumference of the base, and L the slant height, of a cone of revolution.

To prove $\qquad S = C \times \frac{1}{2} L.$

Circumscribe about the cone a regular pyramid; let S denote its lateral area, and C' the perimeter of its base.

Then since the sides of the base of the pyramid are bisected at their points of contact, the slant height of the pyramid is the same as the slant height of the cone. (§ 515.)

Therefore, $\qquad S' = C' \times \frac{1}{2} L.$ (§ 523.)

Now let the number of faces of the pyramid be indefinitely increased.

Then, $\qquad S'$ approaches the limit S. (§ 667, 1.)

And $\quad C' \times \frac{1}{2} L$ approaches the limit $C \times \frac{1}{2} L$. (§ 363, I.)

Therefore, $\qquad S = C \times \frac{1}{2} L.$ (§ 188.)

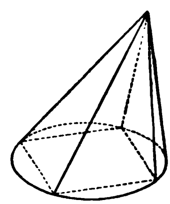
670. Cor. Let S denote the lateral area, T the total area, L the slant height, and R the radius of the base, of a cone of revolution.

Then, $S = 2\pi R \times \frac{1}{2} L$ (§ 368) $= \pi RL$.

And $T = \pi RL + \pi R^2$ (§ 371) $= \pi R(L + R)$.

Proposition V. Theorem.

671. *The volume of a circular cone is equal to one-third the product of its base and altitude.*



Let V denote the volume, B the area of the base, and H the altitude, of a circular cone.

To prove $V = \frac{1}{3} B \times H$.

Inscribe in the cone a pyramid whose base is a regular polygon; let V' denote its volume, and B' the area of its base.

Then, $V' = \frac{1}{3} B' \times H$. (§ 527.)

Now let the number of faces of the pyramid be indefinitely increased.

Then, V' approaches the limit V. (§ 667, 2.)

And $\frac{1}{3} B' \times H$ approaches the limit $\frac{1}{3} B \times H$. (§ 363, II.)

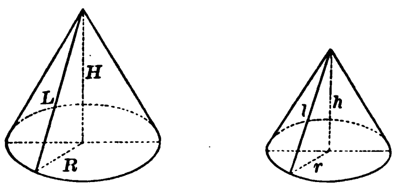
Therefore, $V = \frac{1}{3} B \times H$. (§ 188.)

672. Cor. Let V denote the volume, H the altitude, and R the radius of the base, of a circular cone.

Then, $V = \frac{1}{3} \pi R^2 H$. (§ 371.)

PROPOSITION VI. THEOREM.

673. *The lateral or total areas of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their slant heights, or as the cubes of their altitudes, or as the cubes of the radii of their bases.*



Let S and s denote the lateral areas, T and t the total areas, V and v the volumes, L and l the slant heights, H and h the altitudes, and R and r the radii of the bases, of two similar cones of revolution (§ 568).

To prove $$\frac{S}{s} = \frac{T}{t} = \frac{L^2}{l^2} = \frac{H^2}{h^2} = \frac{R^2}{r^2},$$

and $$\frac{V}{v} = \frac{L^3}{l^3} = \frac{H^3}{h^3} = \frac{R^3}{r^3}.$$

The generating triangles are similar.

Whence, $$\frac{L}{l} = \frac{R}{r} = \frac{H}{h} = \frac{L+R}{l+r}.$$

Then,

$$\frac{S}{s} = \frac{\pi RL}{\pi rl} \quad (\S\ 670) = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{L^2}{l^2} = \frac{H^2}{h^2};$$

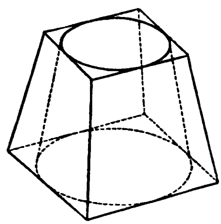
$$\frac{T}{t} = \frac{\pi R\,(L+R)}{\pi r\,(l+r)} \quad (\S\ 670) = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{L^2}{l^2} = \frac{H^2}{h^2};$$

and

$$\frac{V}{v} = \frac{\frac{1}{3}\pi R^2H}{\frac{1}{3}\pi r^2h} \quad (\S\ 672) = \frac{R^2}{r^2} \times \frac{R}{r} = \frac{R^3}{r^3} = \frac{L^3}{l^3} = \frac{H^3}{h^3}.$$

PROPOSITION VII. THEOREM.

674. *The lateral area of a frustum of a cone of revolution is equal to one-half the sum of the circumferences of its bases, multiplied by its slant height.*



Let S denote the lateral area, C and c the circumferences of the bases, and L the slant height, of a frustum of a cone of revolution.

To prove $\qquad S = \frac{1}{2}(C + c) \times L.$

Circumscribe about the frustum a frustum of a regular pyramid.

Let S' denote its lateral area, and C' and c' the perimeters of its bases.

Then since the sides of the bases of the frustum of the pyramid are bisected at their points of contact, the slant height of the frustum of the pyramid is the same as the slant height of the frustum of the cone. (§ 515.)

Therefore, $\qquad S' = \frac{1}{2}(C' + c') \times L.$ (§ 524.)

Now let the number of faces of the frustum of the pyramid be indefinitely increased.

Then, $\qquad S'$ approaches the limit S. (§ 668, 1.)

And

$\frac{1}{2}(C' + c') \times L$ approaches the limit $\frac{1}{2}(C + c) \times L$. (§ 363, I.)

Therefore, $\qquad S = \frac{1}{2}(C + c) \times L.$ (§ 188.)

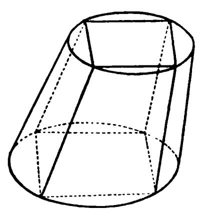
675. COR. I. *The lateral area of a frustum of a cone of revolution is equal to the circumference of a section equally distant from its bases, multiplied by its slant height.*

676. COR. II. Let S denote the lateral area, L the slant height, and R and r the radii of the bases, of a frustum of a cone of revolution.

Then, $S = \frac{1}{2}(2\pi R + 2\pi r) \times L$ (§ 368) $= \pi(R + r) L$.

PROPOSITION VIII. THEOREM.

677. *The volume of a frustum of a circular cone is equal to the sum of its bases and a mean proportional between its bases, multiplied by one-third its altitude.*



Let V denote the volume, B and b the areas of the bases, and H the altitude, of a frustum of a circular cone.

To prove $\quad V = (B + b + \sqrt{B \times b}) \times \frac{1}{3} H.$

Inscribe in the frustum a frustum of a pyramid whose base is a regular polygon; let V' denote its volume, and B' and b' the areas of its bases.

Then, $\quad V' = (B' + b' + \sqrt{B' \times b'}) \times \frac{1}{3} H.$ (§ 532.)

Now let the number of faces of the frustum of the pyramid be indefinitely increased.

Then, $\quad V'$ approaches the limit V. (§ 668, 2.)

And $(B' + b' + \sqrt{B' \times b'}) \times \frac{1}{3} H$ approaches the limit

$(B + b + \sqrt{B \times b}) \times \frac{1}{3} H.$ (§ 363, II.)

Whence, $\quad V = (B + b + \sqrt{B \times b}) \times \frac{1}{3} H.$ (§ 188.)

678. COR. Let V denote the volume, H the altitude, and R and r the radii of the bases, of a frustum of a circular cone.

Then, $B = \pi R^2$, and $b = \pi r^2$. (§ 371.)

Whence, $\sqrt{B \times b} = \sqrt{\pi^2 R^2 r^2} = \pi Rr$.

Therefore, $V = (\pi R^2 + \pi r^2 + \pi Rr) \times \frac{1}{3} H$

$= \frac{1}{3} \pi (R^2 + r^2 + Rr) H.$

EXERCISES.

1. Find the lateral area, total area, and volume of a cylinder of revolution, the diameter of whose base is 18, and whose altitude is 16.

2. Find the lateral area, total area, and volume of a cone of revolution, the radius of whose base is 7, and whose slant height is 25.

3. Find the lateral area, total area, and volume of a frustum of a cone of revolution, the diameters of whose bases are 16 and 6, and whose altitude is 12.

4. Find the altitude and diameter of the base of a cylinder of revolution, whose lateral area is 168π and volume 504π.

5. Find the volume of a cone of revolution, whose slant height is 29 and lateral area 580π.

6. Find the lateral area of a cone of revolution, whose volume is 320π and altitude 15.

7. Find the volume of a cylinder of revolution, whose total area is 170π and altitude 12.

8. The altitude of a cone of revolution is 27, and the radius of its base is 16. What is the diameter of the base of an equivalent cylinder of revolution, whose altitude is 16 ?

9. The area of the entire surface of a frustum of a cone of revolution is 306π, and the radii of its bases are 11 and 5. Find its lateral area and volume.

10. The volume of a frustum of a cone of revolution is 6020π, its altitude is 60, and the radius of its lower base is 15. Find the radius of its upper base and its lateral area.

11. Find the altitude and lateral area of a cone of revolution, whose volume is 800π, and whose slant height is to the diameter of its base as 13 to 10.

THE SPHERE.

DEFINITIONS.

679. A *zone* is a portion of the surface of a sphere included between two parallel planes.

The circumferences of the circles which bound the zone are called the *bases,* and the perpendicular distance between their planes the *altitude.*

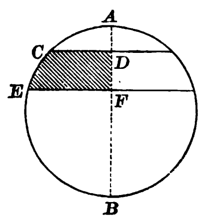
A *zone of one base* is a zone one of whose bounding planes is tangent to the sphere.

680. A *spherical segment* is a portion of a sphere included between two parallel planes.

The circles which bound it are called the *bases,* and the perpendicular distance between them the *altitude.*

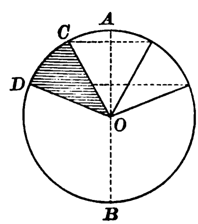
A *spherical segment of one base* is a spherical segment one of whose bounding planes is tangent to the sphere.

681. If the semicircle $ACEB$ be revolved about its diameter AB as an axis, and CD and EF are perpendicular to AB, the arc CE generates a zone whose altitude is DF, and the figure $CEFD$ a spherical segment whose altitude is DF.



The arc AC generates a zone of one base, and the figure ACD a spherical segment of one base.

682. If a semicircle be revolved about its diameter as an axis, any sector generates a solid called a *spherical sector.*

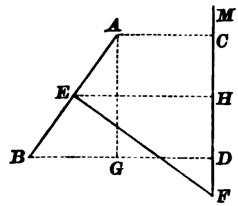


Thus, if the semicircle $ACDB$ be revolved about AB as an axis, the sector OCD generates a spherical sector.

The zone generated by the arc CD is called the *base* of the spherical sector.

PROPOSITION IX. THEOREM.

683. *The area generated by the revolution of a straight line about an axis in its plane, not parallel to itself, is equal to its projection on the axis, multiplied by the circumference of a circle, whose radius is the perpendicular erected at the middle point of the line and terminating in the axis.*



Let the straight line AB be revolved about the axis FM in its plane, not parallel to itself.

Let CD be the projection of AB on FM, and EF the perpendicular erected at the middle point of AB, terminating in the axis.

To prove area $AB^* = CD \times 2\pi EF$. (§ 368.)

Draw AG perpendicular to BD, and EH perpendicular to CD.

The surface generated by AB is the lateral surface of a frustum of a cone of revolution, whose bases are generated by AC and BD.

Then, area $AB = AB \times 2\pi EH$. (§ 675.)

But the triangles ABG and EFH are similar. (§ 261.)

Whence,
$$\frac{AB}{AG} = \frac{EF}{EH}.$$

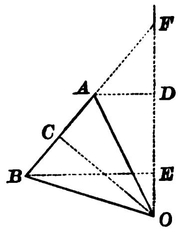
Therefore,
$$AB \times EH = AG \times EF \quad (§ 231.)$$
$$= CD \times EF.$$

Then, area $AB = CD \times 2\pi EF$.

* The expression "area AB" is used to denote the *area generated by AB*.

PROPOSITION X. THEOREM.

684. *If an isosceles triangle be revolved about an axis in its plane, not parallel to its base, which passes through its vertex without intersecting its surface, the volume generated is equal to the area generated by the base, multiplied by one-third the altitude.*



Let the isosceles triangle OAB be revolved about the axis OF in its plane, not parallel to AB; and draw the altitude OC.

To prove vol. OAB* $=$ area $AB \times \frac{1}{3} OC$.

Draw AD and BE perpendicular to OF; and produce BA to meet OF at F.

Now, vol. $OBF =$ vol. $OBE +$ vol. BEF

$= \frac{1}{3}\pi\overline{BE}^2 \times OE + \frac{1}{3}\pi\overline{BE}^2 \times EF$ (§ 672.)

$= \frac{1}{3}\pi\overline{BE}^2 \times (OE + EF) = \frac{1}{3}\pi\overline{BE}^2 \times OF.$

But $BE \times OF = OC \times BF$; for each expresses twice the area of the triangle OBF. (§ 313.)

Hence, vol. $OBF = \frac{1}{3}\pi BE \times OC \times BF.$

But $\pi BE \times BF$ is the area generated by BF. (§ 670.)

Whence, vol. $OBF =$ area $BF \times \frac{1}{3} OC$. (1)

Also, vol. $OAF =$ area $AF \times \frac{1}{3} OC$. (2)

Subtracting (2) from (1), we have

vol. $OAB =$ (area $BF -$ area AF) $\times \frac{1}{3} OC$

$=$ area $AB \times \frac{1}{3} OC$.

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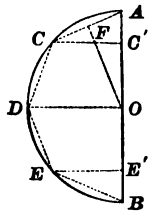
* The expression "vol. OAB" is used to denote the volume generated by OAB.

Proposition XI. Theorem.

685. *The area of the surface of a sphere is equal to its diameter multiplied by the circumference of a great circle.*



Let O be the centre of the semicircle ADB; and let the sphere be generated by the revolution of ADB about AB as an axis.

Let R denote the radius of the sphere.

To prove that the area of the surface of the sphere is

$$AB \times 2\pi R.$$

Divide the arc ADB into four equal arcs, AC, CD, DE, and EB; and draw the chords AC, CD, DE, and EB.

Also, draw CC', DO, and EE' perpendicular to AB, and OF perpendicular to AC.

Then, area $AC = AC' \times 2\pi OF$. (§ 683.)

Also, area $CD = C'O \times 2\pi OF$; etc.

Adding these equations, we have

area generated by broken line $ACDEB$

$$= (AC' + C'O + \text{etc.}) \times 2\pi OF = AB \times 2\pi OF.$$

Now let the subdivisions of the arc ADB be bisected indefinitely.

Then the area generated by the broken line $ACDEB$ approaches the area generated by the arc ADB as a limit. (§ 363, I.)

And $AB \times 2\pi OF$ approaches $AB \times 2\pi R$ as a limit. (§ 364, 1.)

Then area generated by arc $ADB = AB \times 2\pi R$. (§ 188.)

686. Cor. I. Let S denote the area of the surface of a sphere, R its radius, and D its diameter.

Then, $$S = 2\,R \times 2\,\pi R = 4\,\pi R^2.$$

That is, *the area of the surface of a sphere is equal to the square of its radius multiplied by* $4\,\pi$.

Again, $$S = \pi \times (2\,R)^2 = \pi D^2.$$

That is, *the area of the surface of a sphere is equal to the square of its diameter multiplied by* π.

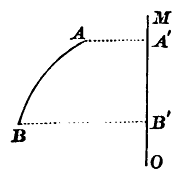
687. Cor. II. Let S and S' denote the areas of the surfaces of two spheres, R and R' their radii, and D and D' their diameters.

Then, $$\frac{S}{S'} = \frac{4\,\pi R^2}{4\,\pi R'^2} = \frac{R^2}{R'^2},$$

and $$\frac{S}{S'} = \frac{\pi D^2}{\pi D'^2} = \frac{D^2}{D'^2}. \quad (\S\ 686.)$$

That is, *the areas of the surfaces of two spheres are to each other as the squares of their radii, or as the squares of their diameters.*

688. Cor. III. Let O be the centre of the arc AB; and draw AA' and BB' perpendicular to the diameter OM.



It may be proved, as in § 685, that the area generated by the revolution of the arc AB about OM as an axis is equal to
$$A'B' \times 2\,\pi R,$$
where R is the radius of the arc.

That is, *the area of a zone is equal to its altitude multiplied by the circumference of a great circle.*

EXERCISES.

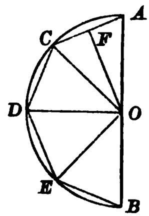
12. Find the area of the surface of a sphere whose radius is 12.

13. Find the area of a zone whose altitude is 13, if the radius of the sphere is 16.

14. The surface of a sphere is equivalent to four great circles.

PROPOSITION XII. THEOREM.

689. ***The volume of a sphere is equal to the area of its surface multiplied by one-third its radius.***



Let O be the centre of the semicircle ADB; and let the sphere be generated by the revolution of ADB about AB as an axis.

Let R denote the radius of the sphere.

To prove that the volume of the sphere is equal to the area of its surface, multiplied by $\frac{1}{3}R$.

Divide the arc ADB into four equal arcs, AC, CD, DE, and EB; and draw the chords AC, CD, DE, and EB.

Draw OC, OD, and OE; also, OF perpendicular to AC.

Then, vol. OAC = area $AC \times \frac{1}{3} OF$. (§ 684.)

Also, vol. OCD = area $CD \times \frac{1}{3} OF$; etc.

Adding these equations, we have

volume generated by polygon $ACDEB$

$$= (\text{area } AC + \text{area } CD + \text{etc.}) \times \tfrac{1}{3} OF$$
$$= \text{area } ACDEB \times \tfrac{1}{3} OF.$$

Now let the subdivisions of the arc ADB be bisected indefinitely.

Then the volume generated by the polygon $ACDEB$ approaches the volume generated by the semicircle ADB as a limit. (§ 363, II.)

And the area generated by $ACDEB \times \frac{1}{3} OF$ approaches the area generated by the arc $ADB \times \frac{1}{3} R$ as a limit.

(§§ 363, I., 364, 1.)

Then volume generated by ADB

$$= \text{area generated by arc } ADB \times \tfrac{1}{3} R. \qquad (\S\ 188.)$$

690. Cor. I. Let V denote the volume of a sphere, R its radius, and D its diameter.

Then, $V = 4\pi R^2 \times \frac{1}{3} R$ (§ 686) $= \frac{4}{3}\pi R^3$.

That is, *the volume of a sphere is equal to the cube of its radius multiplied by* $\frac{4}{3}\pi$.

Again, $V = \frac{1}{6}\pi \times (2R)^3 = \frac{1}{6}\pi D^3$.

That is, *the volume of a sphere is equal to the cube of its diameter multiplied by* $\frac{1}{6}\pi$.

691. Cor. II. Let V and V' denote the volumes of two spheres, R and R' their radii, and D and D' their diameters.

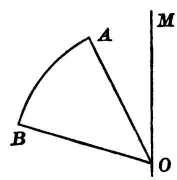
Then, $$\frac{V}{V'} = \frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi R'^3} = \frac{R^3}{R'^3},$$

and $$\frac{V}{V'} = \frac{\frac{1}{6}\pi D^3}{\frac{1}{6}\pi D'^3} = \frac{D^3}{D'^3}. \qquad (\S\ 690.)$$

That is, *the volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

692. Cor. III. Let OAB be a sector of a circle.

It may be proved, as in § 689, that the volume generated by the revolution of OAB about the diameter OM as an axis, is equal to the area generated by the arc AB, multiplied by $\frac{1}{3} R$, where R is the radius of the arc.



That is, *the volume of a spherical sector is equal to the area of the zone which forms its base, multiplied by one-third the radius of the sphere.*

693. Cor. IV. Let h denote the altitude of the zone which forms the base of the spherical sector of § 692.

Then, $$\text{vol. } OAB = h \times 2\pi R \times \tfrac{1}{3} R \qquad (\S\ 688.)$$
$$= \tfrac{2}{3}\pi R^2 h.$$

694. Cor. V. Let P denote the volume of a spherical pyramid, and K the area of its base; also, let V denote the volume, S the area of the surface, and R the radius, of the sphere.

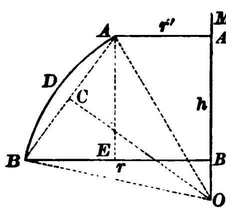
Then, $\frac{P}{K} = \frac{V}{S}$ (§ 657) $= \frac{\frac{4}{3}\pi R^3}{4\pi R^2}$ (§§ 686, 690) $= \frac{1}{3}R$.

Whence, $P = K \times \frac{1}{3}R$.

That is, *the volume of a spherical pyramid is equal to the area of its base multiplied by one-third the radius of the sphere.*

Proposition XIII. Problem.

695. *To find the volume of a spherical segment.*



Let O be the centre of the arc ADB.

Draw AA' and BB' perpendicular to the diameter OM.

To find the volume of the spherical segment generated by the revolution of the figure $ADBB'A'$ about OM as an axis.

Let $AA' = r'$, $BB' = r$, $A'B' = h$, and $OA = R$.

Draw OA, OB, and AB; also, draw OC perpendicular to AB, and AE perpendicular to BB'.

Now, vol. $ADBB'A'$ = vol. $ACBD$ + vol. $ABB'A'$. (1)

Also, vol. $ACBD$ = vol. $OADB$ − vol. OAB.

But, vol. $OADB = \frac{2}{3}\pi R^2 h$. (§ 693.)

And vol. OAB = area $AB \times \frac{1}{3} OC$ (§ 684.)

$= h \times 2\pi OC \times \frac{1}{3} OC$ (§ 683.)

$= \frac{2}{3}\pi \overline{OC}^2 h$.

Then, $\quad \text{vol. } ACDB = \frac{2}{3}\pi R^2 h - \frac{2}{3}\pi \overline{OC}^2 h$

$$= \frac{2}{3}\pi (R^2 - \overline{OC}^2)\, h.$$

But, $\quad R^2 - \overline{OC}^2 = \overline{AC}^2 \quad$ (§ 274.)

$$= (\tfrac{1}{2} AB)^2 = \tfrac{1}{4}\overline{AB}^2.$$

Then, $\quad \text{vol. } ACDB = \frac{2}{3}\pi \times \frac{1}{4}\overline{AB}^2 \times h = \frac{1}{6}\pi \overline{AB}^2 h.$

Now, $\quad \overline{AB}^2 = \overline{BE}^2 + \overline{AE}^2 \quad$ (§ 273.)

$$= (r - r')^2 + h^2.$$

Then, $\quad \text{vol. } ACDB = \frac{1}{6}\pi\,[(r - r')^2 + h^2]\, h.$

Also, $\quad \text{vol. } ABB'A' = \frac{1}{3}\pi (r^2 + r'^2 + rr')\, h. \quad$ (§ 678.)

Substituting in (1), we have

$$\begin{aligned}
&\text{vol. } ADBB'A' \\
&= \tfrac{1}{6}\pi\,[(r - r')^2 + h^2]\, h + \tfrac{1}{3}\pi (r^2 + r'^2 + rr')\, h \\
&= \tfrac{1}{6}\pi (r^2 - 2\,rr' + r'^2 + h^2 + 2\,r^2 + 2\,r'^2 + 2\,rr')\, h \\
&= \tfrac{1}{6}\pi (3\,r^2 + 3\,r'^2 + h^2)\, h \\
&= \tfrac{1}{2}\pi (r^2 + r'^2)\, h + \tfrac{1}{6}\pi h^3.
\end{aligned}$$

696. Cor. If r denotes the radius of the base, and h the altitude, of a spherical segment of one base, its volume is

$$\tfrac{1}{2}\pi r^2 h + \tfrac{1}{6}\pi h^3.$$

EXERCISES.

15. Find the volume of a sphere whose radius is 12.

16. Find the volume of a spherical sector the altitude of whose base is 12, the diameter of the sphere being 25.

17. Find the volume of a spherical segment, the radii of whose bases are 4 and 5, and whose altitude is 9.

18. Find the radius and volume of a sphere the area of whose surface is 324π.

19. Find the diameter and area of the surface of a sphere whose volume is $\frac{1125}{2}\pi$.

20. The surface of a sphere is equivalent to the lateral surface of its circumscribed cylinder.

21. The volume of a sphere is two-thirds the volume of its circumscribed cylinder.

22. A spherical cannon-ball 9 in. in diameter is dropped into a cubical box filled with water, whose depth is 9 in. How many cubic inches of water will be left in the box?

23. What is the angle of the base of a spherical wedge whose volume is $\frac{40}{3}\pi$, if the radius of the sphere is 4?

24. Find the area of a spherical triangle whose angles are 125°, 133°, and 156°, on a sphere whose radius is 10.

25. Find the volume of a quadrangular spherical pyramid, the angles of whose base are 107°, 118°, 134°, and 146°; the diameter of the sphere being 12.

26. The surface of a sphere is equivalent to two-thirds the entire surface of its circumscribed cylinder.

27. The volume of a cylinder of revolution is equal to its lateral area multiplied by one-half the radius of its base.

28. Prove Prop. IX. when the straight line is parallel to the axis.

29. Find the area of the surface and the volume of a sphere, inscribed in a cube the area of whose surface is 486.

30. How many spherical bullets, each $\frac{5}{8}$ in. in diameter, can be formed from five pieces of lead, each in the form of a cone of revolution, the radius of whose base is 5 in., and whose altitude is 8 in.?

31. Find the volume of a sphere circumscribing a cube whose volume is 64.

32. A cylindrical vessel, 8 in. in diameter, is filled to the brim with water. A ball is immersed in it, displacing water to the depth of $2\frac{1}{4}$ in. Find the diameter of the ball.

33. The radii of the bases of two similar cylinders of revolution are 24 and 44, respectively. If the lateral area of the first cylinder is 720, what is the lateral area of the second?

34. The slant heights of two similar cones of revolution are 9 and 15, respectively. If the volume of the second cone is 625, what is the volume of the first?

35. The total areas of two similar cylinders of revolution are 32 and 162, respectively. If the volume of the second cylinder is 1458, what is the volume of the first?

36. The volumes of two similar cones of revolution are 343 and 512, respectively. If the lateral area of the first cone is 196, what is the lateral area of the second?

37. If a sphere 6 in. in diameter weighs 351 ounces, what is the weight of a sphere of the same material whose diameter is 10 in.?

38. If a sphere whose radius is $12\frac{1}{2}$ in. weighs 3125 lb., what is the radius of a sphere of the same material whose weight is $819\frac{1}{5}$ lb.?

39. The altitude of a frustum of a cone of revolution is $3\frac{1}{2}$, and the radii of its bases are 5 and 3. What is the diameter of an equivalent sphere?

40. Find the radius of a sphere whose surface is equivalent to the entire surface of a cylinder of revolution, whose altitude is $10\frac{1}{2}$ and radius of base 3.

41. A cubical piece of lead, the area of whose entire surface is 384 sq. in., is melted and formed into a cone of revolution, the radius of whose base is 12 in. Find the altitude of the cone.

42. The volume of a cylinder of revolution is equal to the area of its generating rectangle, multiplied by the circumference of a circle whose radius is the distance to the axis from the centre of the rectangle.

43. The volume of a cone of revolution is equal to its lateral area, multiplied by one-third the distance of any element of its lateral surface from the centre of the base.

44. Two zones on the same sphere, or on equal spheres, are to each other as their altitudes.

45. The area of a zone of one base is equal to the area of the circle whose radius is the chord of its generating arc.

46. If the radius of a sphere is R, what is the area of a zone of one base, whose generating arc is $45°$?

47. If the altitude of a cone of revolution is 24, and its slant height 25, find the total area of an inscribed cylinder, the radius of whose base is 2.

48. Find the area of the surface and the volume of a sphere circumscribing a cylinder of revolution, the radius of whose base is 9, and whose altitude is 24.

49. A cone of revolution is circumscribed about a sphere whose diameter is two-thirds the altitude of the cone. Prove that its lateral surface and volume are, respectively, three-halves and nine-fourths the surface and volume of the sphere.

50. If the altitude of a cone of revolution is three-fourths the radius of its base, its volume is equal to its lateral area multiplied by one-fifth the radius of its base.

51. A cone of revolution is inscribed in a sphere whose diameter is $\frac{4}{3}$ the altitude of the cone. Prove that its lateral surface and volume are, respectively, $\frac{3}{8}$ and $\frac{9}{32}$ the surface and volume of the sphere.

52. If the radius of a sphere is 25, find the lateral area and volume of an inscribed cone, the radius of whose base is 24.

53. If the volume of a sphere is $\frac{500}{3}\pi$, find the lateral area and volume of a circumscribed cone whose altitude is 18.

54. Find the volume of a spherical segment of one base whose altitude is 6, the diameter of the sphere being 30.

55. A circular sector whose central angle is 45° and radius 12, revolves about a diameter perpendicular to one of its bounding radii. Find the volume of the spherical sector generated.

56. Two equal small circles of a sphere are equally distant from the centre.

57. A square whose area is A, revolves about its diagonal as an axis. Find the area of the entire surface, and the volume, of the solid generated.

58. How many cubic feet of metal are there in a hollow cylindrical tube 18 ft. long, whose outer diameter is 8 in., and thickness 1 in.?

59. The altitude of a cone of revolution is 9 in. At what distances from the vertex must it be cut by planes parallel to its base, in order that it may be divided into three equivalent parts?

60. Given the radius of the base, R, and the total area, T, of a cylinder of revolution, to find its volume.

61. Given the diameter of the base, D, and the volume, V, of a cylinder of revolution, to find its lateral area and total area.

62. Given the altitude, H, and the volume, V, of a cone of revolution, to find its lateral area.

63. Given the slant-height, L, and the lateral area, S, of a cone of revolution, to find its volume.

64. Given the area of the surface of a sphere, S, to find its volume.

65. Given the volume of a sphere, V, to find the area of its surface.

66. A right triangle whose legs are a and b revolves about its hypotenuse as an axis. Find the area of the entire surface, and the volume, of the solid generated.

67. The parallel sides of a trapezoid are 12 and 33, respectively, and its non-parallel sides are 10 and 17. Find the volume generated by the revolution of the trapezoid about its longest side as an axis.

68. Find the diameter of a sphere in which the area of the surface and the volume are expressed by the same numbers.

69. An equilateral triangle, whose side is a, revolves about one of its sides as an axis. Find the area of the entire surface, and the volume, of the solid generated.

70. An equilateral triangle, whose altitude is h, revolves about one of its altitudes as an axis. Find the area of the surface, and the volume, of the solids generated by the triangle, and by its inscribed circle.

71. Find the lateral area and volume of a cylinder of revolution, whose altitude is equal to the diameter of its base, inscribed in a cone of revolution whose altitude is h, and radius of base r.

72. Find the lateral area and volume of a cylinder of revolution, whose altitude is equal to the diameter of its base, inscribed in a sphere whose radius is r.

73. An equilateral triangle, whose side is a, revolves about a straight line drawn through one of its vertices parallel to the opposite side. Find the area of the entire surface, and the volume, of the solid generated.

74. If the radius of a sphere is R, find the circumference and area of a small circle, whose distance from the centre is h.

75. The outer diameter of a spherical shell is 9 in., and its thickness is 1 in. What is its weight, if a cubic inch of the metal weighs $\frac{1}{3}$ lb.?

76. A regular hexagon, whose side is a, revolves about its longest diagonal as an axis. Find the area of the entire surface, and the volume, of the solid generated.

77. The sides AB and BC of a rectangle $ABCD$ are 5 and 8, respectively. Find the volumes generated by the revolution of the triangle ACD about the sides AB and BC as axes.

78. The sides of a triangle are 17, 25, and 28. Find the volume generated by the revolution of the triangle about its longest side as an axis.

79. The cross-section of a tunnel $2\frac{1}{2}$ miles in length is in the form of a rectangle 6 yd. wide and 4 yd. high, surmounted by a semicircle whose diameter is equal to the width of the rectangle. How many cubic yards of material were taken out in its construction?

80. A frustum of a circular cone is equivalent to three cones, whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum. (§ 677.)

81. The volume of a cone of revolution is equal to the area of its generating triangle, multiplied by the circumference of a circle whose radius is the distance to the axis from the intersection of the medians of the triangle.

82. If the earth be regarded as a sphere whose radius is R, what is the area of the zone visible from a point whose height above the surface is H?

83. The sides AB and BC of an acute-angled triangle ABC, are $\sqrt{241}$ and 10, respectively. Find the volume generated by the revolution of the triangle about an axis in its plane, not intersecting its surface, whose distances from A, B, and C are 2, 17, and 11, respectively.

84. A projectile consists of two hemispheres, connected by a cylinder of revolution. If the altitude and diameter of the base of the cylinder are 8 in. and 7 in., respectively, find the number of cubic inches in the projectile.

85. A tapering hollow iron column, 1 in. thick, is 24 ft. long, 10 in. in outside diameter at one end, and 8 in. in diameter at the other. How many cubic inches of metal were used in its construction?

86. If any triangle be revolved about an axis in its plane, not parallel to its base, which passes through its vertex without intersecting its surface, the volume generated is equal to the area generated by the base, multiplied by one-third the altitude.

87. If any triangle be revolved about an axis which passes through its vertex parallel to its base, the volume generated is equal to the area generated by the base, multiplied by one-third the altitude.

88. A segment of a circle whose bounding arc is a quadrant, and whose radius is r, revolves about a diameter parallel to its bounding chord. Find the area of the entire surface, and the volume, of the solid generated.

89. Find the area of the surface of the sphere circumscribing a regular tetraedron whose edge is 8.

ANSWERS

TO THE

NUMERICAL EXERCISES.

NOTE. **Those answers are omitted which, if given, would destroy the utility of the problem.**

Page 16.

6. 24°. **7.** 63° 30′, 26° 30′. **8.** 22° 30′, 157° 30′.

9. 37°.

Page 41.

27. $A = 20°$, $B = 60°$, $C = 100°$.

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31. $A = 112° 30'$, $B = 33° 45'$, $C = 33° 45'$.

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96. 7.

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15. 52° 30′. **17.** 96°. **18.** 164°.

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21. 28° 45′. **22.** 44° 30′. **23.** 12°.

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25. 54° 30′. **26.** 178°.

Page 101.

29. 112° 30′. **42.** 83°, 89° 30′, 97°, 90° 30′, 74° 30′.

Page 102.

55. $\angle AED = 14° 30'$, $\angle AFB = 10° 30'$.

58. 114° 30′, 89° 30′, 65° 30′, 90° 30′.

Page 103.

70. 97° 30′, 89° 30′, 82° 30′, 90° 30′.

Page 126.

1. 112. 2. 42. 3. $\frac{25}{27}$. 4. 63.

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5. $3\frac{1}{6}$, $2\frac{4}{5}$. 6. $11\frac{2}{3}$, $18\frac{2}{3}$.

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7. $19\frac{3}{5}$, $25\frac{1}{5}$.

Page 141.

9. 4 ft. 6 in. 10. 12.

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12. 15. 13. 37 ft. 1 in. 14. 47 ft. 6 in. 15. 4.33 +.
16. 1 ft. 9.06 + in.

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18. 58. 19. 24.

Page 153.

21. 21. 24. 26. 27. 48. 28. 10. 29. $13\frac{1}{3}$.
30. 12.726 +. 31. 45.

Page 154.

33. $17\frac{2}{3}$. 36. 50. 40. $\sqrt{129}$, $2\sqrt{21}$, $\sqrt{201}$. 41. $3\frac{1}{3}$.

Page 155.

46. 36. 48. 63. 49. 3 and 4; $1\frac{4}{5}$ and $3\frac{1}{5}$. 55. 24.
56. 17. 57. 21, 28. 58. $8\sqrt{3}$. 59. 12, 4.

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61. $3\sqrt{3}$. 62. 14. 69. 21. 72. 70 and 99; 65 and 117.

Page 165.

1. 4 : 3.

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2. $30\frac{5}{8}$ ft. 3. 8 ft. 9 in. 4. 14, 12. 5. 6 ft. 11 in., 20 ft. 9 in.
6. 6 ft. 7 in.

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7. 6 sq. ft. 60 sq. in.

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8. 2 sq. ft. 48 sq. in. 9. 243.

Page 175.

11. 210; $16\frac{4}{5}$, $24\frac{12}{17}$, 15. 12. 73. 13. 117.

Page 177.

18. $\frac{25}{4}\sqrt{3}$. 19. $3\sqrt{3}$. 20. 34. 22. 120. 25. 210.
26. 18.

Page 178.

27. $1\frac{1}{2}$ ft. 28. 6. 29. $4\sqrt{3}$. 30. 1260. 34. 120. 35. 17.
36. 624. 38. 540. 39. 28. 42. $4\frac{1}{3}$. 43. 30, 16.

Page 179.

44. $36\frac{1}{4}$. 47. $\frac{15}{2}\sqrt{2}$, $11\sqrt{2}$. 48. 54. 51. 39, 45. 52. 1010.
53. 336.

Page 207.

29. 9. 30. 88 : 121. 31. 13. 32. $\frac{9}{2}\sqrt{2}$.

Page 211.

34. 15.708, 19.635. 35. Area, 452.3904.
36. Circumference, 50.2656. 49. 35.6048. 50. 35.81424.
51. 9.827.

Page 212.

52. 10.2102. 53. 72. 54. 150.7968. 55. 1.2732.
56. 201.0624. 57. 18.8496. 58. 50.2656.
59. 37.6992, 9.4248. 60. 25.1328, 35.5377. 61. 9.06.
62. 1306.9056 sq. ft. 63. 120.99 ft. 64. 57 in.
65. 57.295°+. 66. 2.658. 67. 5.64.

Page 224.

55. $10\sqrt{7}$.

Page 225.

61. 6, 8. 65. $1\frac{13}{16}$.

Page 228.

91. 480.

Page 275.

1. 4:3. 2. 2:5.

Page 277.

4. 42. 5. 1 ft. 9 in. 6. $34\frac{21}{64}$ cu. in.; $63\frac{3}{8}$ sq. in. 7. 574.
8. 1008. 9. 12 and 7. 10. 1944. 13. 17.

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14. Volume, $50\sqrt{3}$.

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16. Volume, $\frac{243}{2}\sqrt{3}$.

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19. $\sqrt{273}$, $18\sqrt{237}$, $180\sqrt{3}$. 20. $\frac{1}{2}\sqrt{118}$, $3\sqrt{109}$, 15.
21. $\sqrt{97}$, $12\sqrt{93}$, $72\sqrt{3}$. 22. $4\sqrt{39}$, $504\sqrt{3}$, $936\sqrt{3}$.
23. $6\sqrt{3}$, $56\sqrt{26}$, $503\frac{1}{3}$. 24. $4\sqrt{10}$, $72\sqrt{39}$, $672\sqrt{3}$.
25. 150. 26. 320. 27. 2400 sq. in. 28. $3\frac{193}{432}$ cu. ft.
29. 770. 30. Volume, $48\sqrt{5}$. 31. 840. 32. 36 sq. in.
33. 12 in.

Page 294.

34. 512, 384. 35. 1705. 36. 144. 37. 10, 1.
38. 700, 1568. 39. $\frac{7}{3}\sqrt{57}$, $640\sqrt{3}$. 40. $42\sqrt{91}$, $624\sqrt{3}$.
41. 240, $\frac{100}{3}\sqrt{119}$. 42. 108, $21\sqrt{39}$. 43. 768, 2340.

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49. 50. 52. $4\sqrt{3}$, $\frac{2}{3}\sqrt{2}$. 53. 15. 59. 582.

Page 296.

65. 9600 lb. 66. $168\sqrt{3}$, $15\sqrt{219}$. 67. 5700 cu. yd.
68. $\frac{15}{4}\sqrt{35}$.

Page 308.

73. 3456 cu. in. 74. 6 ft. 75. 4 ft. 6 in. 76. $5\sqrt[3]{4}$ in.
77. 960, 3072. 78. 128. 79. 12. 80. 6. 84. $36\sqrt{3}$.

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11. 45°. 12. $4\sqrt{3}$, $\sqrt{3}$.

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13. 36. 14. 44. 15. $39\frac{2}{3}$. 16. 86° 24′. 17. 3:2. 18. 108°.
19. 220.

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22. $66\frac{1}{2}$. 23. $36\frac{1}{4}$.

Page 347.

29. 60 cu. ft. 30. 153°. 38. 30 in., 8 in., 20 in.

Page 358.

1. 288π, 450π, 1296π. 2. 175π, 224π, 392π. 3. 143π, 216π, 388π
4. 14, 12. 5. 2800π.
6. 136π. 7. 300π. 8. 24. 9. 160π, 536π.
10. 4, 1159π. 11. 24, 260π.

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12. 576π. 13. 416π.

Page 367.

15. 2304π. 16. 1250π. 17. 306π. 18. Volume, 972π.
19. Area of surface, 225π. 22. 347.2956 cu. in.

Page 368.

23. 56° 15′. 24. 130π. 25. 58π. 29. 81π, $\frac{243}{2}\pi$.
30. 8192. 31. $32\pi\sqrt{3}$. 32. 6 in. 33. 2420.
34. 135. 35. 128. 36. 256. 37. 1625 oz. 38. 8 in.

Page 369.

39. 7. 40. $4\frac{1}{2}$. 41. $\frac{32}{3\pi}$ in. 46. $\pi R^2(2-\sqrt{2})$.
47. $\frac{536}{7}\pi$. 48. 900π, 4500π.

Page 370.

52. 960π, 6144π. 53. $\frac{585}{4}\pi$, $\frac{675}{2}\pi$. 54. 468π.
55. $576\pi\sqrt{2}$. 57. $\pi A\sqrt{2}$, $\frac{1}{6}\pi A\sqrt{2A}$. 58. 2.7489 +.
59. $3\sqrt[3]{9}$ in., $3\sqrt[3]{18}$ in. 60. $\dfrac{RT-2\pi R^3}{2}$.

61. $\dfrac{4V}{D}$, $\dfrac{8V+\pi D^3}{2D}$. 62. $\dfrac{\sqrt{9V^2+3\pi H^3 V}}{H}$.

63. $\dfrac{S^2\sqrt{\pi^2 L^4-S^2}}{3\pi^2 L^3}$. 64. $\dfrac{S\sqrt{S}}{6\sqrt{\pi}}$. 65. $\sqrt[3]{36\pi V^2}$.

66. $\dfrac{\pi(a+b)ab}{\sqrt{a^2+b^2}}$, $\dfrac{\pi a^2b^2}{3\sqrt{a^2+b^2}}$. 67. 1216π.

Page 371.

69. $\pi a^2 \sqrt{3}$, $\frac{1}{4} \pi a^3$.

70. By triangle, πh^2, $\frac{1}{3} \pi h^3$; by inscribed circle, $\frac{4}{9} \pi h^2$, $\frac{4}{81} \pi h^3$.

71. $\dfrac{4 \pi r^2 h^2}{(2r + h)^2}$, $\dfrac{2 \pi r^3 h^3}{(2r + h)^3}$.

72. $2 \pi r^2$, $\frac{1}{2} \pi r^3 \sqrt{2}$. 73. $2 \pi a^2 \sqrt{3}$, $\frac{1}{2} \pi a^3$. 75. 67.3698 + lb.

76. $2 \pi a^2 \sqrt{3}$, πa^3. 77. $\frac{640}{3} \pi$, $\frac{400}{3} \pi$. 78. 2100π.

79. 167803.68.

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82. $\dfrac{2 \pi R^2 H}{R + H}$. 83. 1440π. 84. 487.4716.

85. 7238.2464. 88. $2 \pi r^2 (1 + \sqrt{2})$, $\frac{1}{3} \pi r^3 \sqrt{2}$.

89. 96π.

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